Real and Complex Representations

This note extends Schur's Lemma to real representations of a compact Lie group, expanding on some of the material in §5 of Chapter II in Bröcker–tom Dieck.

Throughout, let G be a compact Lie group. Our object is to compute the endomorphism ring $\operatorname{End}_{\mathbb{R}}(W)$ of an irreducible real representation W by appealing to the complex version of Schur's Lemma.

Two constructions We need two constructions that relate real and complex representations. Given a complex representation V of G, we may regard V as a real vector space (of twice the dimension) and treat it as a real representation of G, the *realification* rV of V. Trivially, $r\overline{V} = rV$.

In the other direction, given a real representation W of G, we have the *complexification* $cW = \mathbb{C} \otimes_{\mathbb{R}} W$, equipped with the obvious actions of $\mathbb{C}$ and G. Then $t \otimes W \colon \overline{cW} \cong cW$, where $t \colon \mathbb{C} \to \mathbb{C}$ denotes the conjugation in $\mathbb{C}$. (The identity morphism does not work!)

In this note, V will always denote a complex representation and W a real representation. We recall that $rcW \cong W \oplus W$ and $crV \cong V \oplus \overline{V}$.

The first half of Schur's Lemma carries over without change (by the same proof).

LEMMA 1 *If W_1 and W_2 are real irreducible representations of G, any morphism $f \colon W_1 \to W_2$ is either an isomorphism or zero.* □

COROLLARY 2 *For any irreducible real representation W of G, the endomorphism ring $\operatorname{End}_{\mathbb{R}}(W)$ is a finite-dimensional skew field over $\mathbb{R}$.* □

Complex vector spaces with conjugation The above space cW is more than just a complex vector space.

DEFINITION 3 A complex vector space X has a *real structure* or *conjugation* if we are given an involution $x \mapsto \bar{x}$ that is additive and antilinear: $\overline{zx} = \bar{z}\,\bar{x}$ for $z \in \mathbb{C}$. Its *real part* $\Re X$ is the real vector subspace consisting of those $x \in X$ that satisfy $\bar{x} = x$.

If X is a complex representation of G, we require the conjugation to preserve the G-action, $\overline{gx} = g\bar{x}$; then $\Re X$ is a real representation of G.

In the case of cW above, the conjugation is inherited from $\mathbb{C}$ and does indeed preserve the G-action; also, $\Re cW = 1 \otimes W \cong W$.

LEMMA 4 *If X is a complex vector space with conjugation, it is canonically isomorphic, as a complex vector space, to $\mathbb{C} \otimes_{\mathbb{R}} \Re X$. Explicitly, it is the direct sum of the real subspaces $\Re X$ and $i\Re X$.*

Proof For any x, $x + \bar{x}$ and $i\bar{x} - ix$ lie in $\Re X$, so $x = (x+\bar{x})/2 + i(i\bar{x} - ix)/2 \in \Re X + i\Re X$. On the other hand, if $x \in (\Re X) \cap (i\Re X)$, we have $x = iy$ with $y \in \Re X$, and $\bar{x} = -iy = -x$. Since $\bar{x} = x$, this implies $x = 0$. □

LEMMA 5 *Let W be any real representation of G. Then:*

(a) *The complexification cW is a complex representation with conjugation;*

(b) *The complex endomorphism ring $\mathrm{End}_{\mathbb{C}}(cW)$ is a complex vector space with conjugation;*

(c) *We may identify the real part $\Re\,\mathrm{End}_{\mathbb{C}}(cW)$ of $\mathrm{End}_{\mathbb{C}}(cW)$ with the real endomorphism ring $\mathrm{End}_{\mathbb{R}}(W)$ of W itself.*

Proof In (a), given a complex endomorphism f of cW, we define its conjugate $\bar{f}$ by

$$\bar{f}v = \overline{f\bar{v}}. \tag{6}$$

It is easy to check that $\bar{f}$ is again $\mathbb{C}$-linear and that this conjugation is antilinear.

In (b), suppose that $f\colon cW \to cW$ is an endomorphism. We decompose $cW = (1\otimes_{\mathbb{R}}W)\oplus(i\otimes_{\mathbb{R}}W)$ and consider the restriction $f|1\otimes W$. We may write $f(1\otimes w) = 1\otimes f_1 w + i\otimes f_2 w$, where f_1 and f_2 are real endomorphisms of W. Then by $\mathbb{C}$-linearity of f, $f(i\otimes w) = -1\otimes f_2 w + i\otimes f_1 w$, which shows that the real endomorphisms f_1 and f_2 determine f and may be chosen arbitrarily. The conjugation simply changes the sign of f_2, so that $\Re f$ is essentially just f_1. □

Remark It will be useful to rewrite the definition and identify the real endomorphisms of W with those complex endomorphisms of cW that commute with the conjugation on cW.

LEMMA 7 *Given a real representation W of G, suppose that V is a self-conjugate invariant complex subspace of cW. Then V has the form $\mathbb{C}\otimes_{\mathbb{R}} W_1$, where W_1 is some invariant real subspace of W.*

Proof We define the invariant subspace W_1 by $1\otimes W_1 = V\cap(1\otimes W)$; since $1\otimes W_1 \subset V$, we have $\mathbb{C}\otimes_{\mathbb{R}} W_1 \subset V$. Conversely, given an element $v = 1\otimes w_1 + i\otimes w_2 \in V$, its conjugate $1\otimes w_1 - i\otimes w_2$ also lies in V, from which it follows that $1\otimes w_1$, $i\otimes w_2$ and $1\otimes w_2$ all lie in V. Thus w_1 and w_2 lie in W_1 and $v \in \mathbb{C}\otimes_{\mathbb{R}} W_1$. □

We use Lemma 5 to compute the desired endomorphism rings directly. Let W be an irreducible real representation. We decompose the complex representation cW into irreducible complex representations; since $rcW \cong W\oplus W$, cW decomposes into either one or two complex representations. There are three cases.

Case 1: $V = cW$ is irreducible. By Schur's Lemma, $\mathrm{End}_{\mathbb{C}}(V)\cong\mathbb{C}$ consists of the endomorphisms $fv = \lambda v$, for $\lambda\in\mathbb{C}$. Then $\bar{f}v = \overline{\lambda\bar{v}} = \bar{\lambda}v$, so that $\bar{f}$ corresponds to $\bar{\lambda}$. As a trivial example, we have the trivial representation $\mathbb{R}$ of any group G.

THEOREM 8 *Let W be an irreducible real representation of G. If cW is irreducible as a complex representation, the endomorphism ring $\mathrm{End}_{\mathbb{R}}(W)\cong\mathbb{R}$ consists of the endomorphisms $fw = \lambda w$ with λ real.* □

Otherwise, we have the decomposition $cW = V_1\oplus V_2$ into exactly two irreducible invariant complex subspaces. By Lemma 7, the only self-conjugate invariant complex subspaces of cW are 0 and cW. In particular, we deduce that $V_1\cap\overline{V}_1 = 0$ and $V_1+\overline{V}_1 = cW$, so that $cW = V_1\oplus\overline{V}_1$. This case in turn divides into two.

Case 2: $cW = V \oplus \overline{V}$, where V is irreducible and not isomorphic to $\overline{V}$. By Schur's Lemma, $\mathrm{End}_{\mathbb{C}}(cW) \cong \mathbb{C} \oplus \mathbb{C}$; the endomorphism corresponding to the pair $(\lambda, \mu) \in \mathbb{C} \oplus \mathbb{C}$ is simply $f_{\lambda,\mu}(v_1, \bar{v}_2) = (\lambda v_1, \mu \bar{v}_2)$. The conjugation in cW is obvious, $\overline{(v_1, \bar{v}_2)} = (v_2, \bar{v}_1)$, and (6) yields

$$\bar{f}_{\lambda,\mu}(v_1, \bar{v}_2) = \overline{f_{\lambda,\mu}(v_2, \bar{v}_1)} = \overline{(\lambda v_2, \mu \bar{v}_1)} = (\bar{\mu} v_1, \bar{\lambda} \bar{v}_2) = f_{\bar{\mu},\bar{\lambda}}(v_1, \bar{v}_2).$$

Thus $f_{\lambda,\mu}$ is real if and only if $\mu = \bar{\lambda}$, and then $\lambda \in \mathbb{C}$ determines $f_{\lambda,\mu}$.

THEOREM 9 *Let W be an irreducible real representation of G. If $cW = V \oplus \overline{V}$, where V is not isomorphic to $\overline{V}$, then* $\mathrm{End}_{\mathbb{R}}(W) \cong \mathbb{C}$. □

As a simple example, we have the standard representation of $SO(2)$ on $\mathbb{R}^2$.

Case 3: $cW = V \oplus \overline{V}$, where V is irreducible and $V \cong \overline{V}$. So $cW \cong V \oplus V$, and by Schur's Lemma, $\mathrm{End}_{\mathbb{C}}(cW) \cong M_2(\mathbb{C})$. The 2×2 complex matrix A with entries $a_{j,k}$ acts on $V \oplus V$ in the usual way,

$$A(v_1, v_2) = (a_{1,1} v_1 + a_{1,2} v_2, a_{2,1} v_1 + a_{2,2} v_2).$$

We need to be specific. We choose some isomorphism $\phi\colon V \cong \overline{V}$, and use it to form the isomorphism $\Phi = \mathrm{id} \oplus \phi\colon V \oplus V \cong V \oplus \overline{V}$. In order to work entirely in V, it will be convenient to write $\phi v = \overline{\theta v}$, where $\theta\colon V \to V$ is an *antilinear* bijection, so that $\Phi(v_1, v_2) = (v_1, \overline{\theta v_2})$.

Now $\theta \circ \theta$ is $\mathbb{C}$-linear, and therefore has the form $\theta\theta v = \lambda v$ for some $\lambda \in \mathbb{C}$ by Schur's Lemma, where $\lambda \neq 0$. Moreover, by writing $\theta \circ \theta \circ \theta$ two different ways, $\theta\theta\theta v = (\theta\theta)\theta v = \lambda \theta v$ and $\theta\theta\theta v = \theta(\theta\theta v) = \theta(\lambda v) = \bar{\lambda}\theta v$, we see that λ must be real. Further, if we replace ϕ by $k\phi$, where k is real, λ changes to $k^2\lambda$, which allows us to arrange $\lambda = \pm 1$ by a suitable choice of ϕ.

The conjugation in $V \oplus \overline{V}$ corresponds by Φ to the conjugation $\overline{(v_1, v_2)} = (\theta v_2, \theta^{-1} v_1)$ in $V \oplus V$. By Lemma 5, the matrix A lies in the real part of $\mathrm{End}_{\mathbb{C}}(V \oplus V)$ if and only if it commutes with conjugation. If we conjugate first and then apply A, we find

$$(a_{1,1}\theta v_2 + a_{1,2}\theta^{-1} v_1, a_{2,1}\theta v_2 + a_{2,2}\theta^{-1} v_1).$$

If we apply A first and then conjugate, we find, using the antilinearity of θ and θ^{-1},

$$(\bar{a}_{2,1}\theta v_1 + \bar{a}_{2,2}\theta v_2, \bar{a}_{1,1}\theta^{-1} v_1 + \bar{a}_{1,2}\theta^{-1} v_2).$$

Comparison of the coefficients of $\theta^{-1} v_1$ in the second entries yields $a_{2,2} = \bar{a}_{1,1}$. To continue, we rewrite $\bar{a}_{2,1}\theta v_1$ as $\bar{a}_{2,1}\lambda\theta^{-1} v_1$, to obtain $a_{1,2} = \lambda \bar{a}_{2,1}$. The terms with v_2 give nothing new. Thus $\Re\,\mathrm{End}_{\mathbb{C}}(V \oplus V)$ consists of those 2×2 complex matrices of the form

$$A = \begin{pmatrix} a & \lambda\bar{b} \\ b & \bar{a} \end{pmatrix}.$$

However, we observe that $\det(A) = |a|^2 - \lambda|b|^2$, which vanishes if $\lambda = 1$ and $a = b$; in view of Corollary 2, we conclude that we must have $\lambda = -1$. We have recovered a standard construction of the quaternions $\mathbb{H}$.

As an obvious example, the group S^3 of unit quaternions acts on $\mathbb{H}$ by left multiplication; the endomorphism corresponding to $q \in \mathbb{H}$ is right multiplication by $\bar{q}$.

THEOREM 10 *Let W be an irreducible real representation of G. If $cW = V \oplus \overline{V}$, where $V \cong \overline{V}$, the endomorphism ring* $\mathrm{End}_{\mathbb{R}}(W)$ *is isomorphic to* $\mathbb{H}$. □

Note that we have constructed the skew field $\mathbb{H}$ of quaternions out of thin air.

Characters One can define the character function $\chi_W(g)$ of a real representation W just as for complex representations; it is of course real-valued. Further, distinct irreducible real representations W_1 and W_2 yield orthogonal characters χ_{W_1} and χ_{W_2}, just as in the complex case. However, they need not be unit vectors in the L^2-norm $\|-\|_2$. In fact, the norm gives an easy way to distinguish the three classes of real representations.

First, we need to record the behavior of the operations r and c on characters.

PROPOSITION 11 *If W is a real representation, $\chi_{cW} = \chi_W$. If V is a complex representation, $\chi_{rV} = 2\Re\chi_V$.*

Proof The action of $g \in G$ on W is described by a real matrix $A(g)$. The action of g on cW is described by the same matrix $A(g)$, reinterpreted as a complex matrix; the trace is unchanged.

Suppose V has the $\mathbb{C}$-basis $(e_1, e_2, \ldots, e_n)$, with the action of g given by the complex matrix $B(g)$. Then V has the real basis $(e_1, ie_1, e_2, ie_2, \ldots, ie_n)$. The effect on $B(g)$ is to replace each complex entry $z = a + ib$ by the 2×2 matrix $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. In computing the trace, each z on the diagonal is replaced by two copies of a. □

THEOREM 12 *Let W be an irreducible real representation.*

(a) *If* $\mathrm{End}_{\mathbb{R}}(W) \cong \mathbb{R}$, *then* $\|\chi_W\|_2 = 1$.

(b) *If* $\mathrm{End}_{\mathbb{R}}(W) \cong \mathbb{C}$, *then* $\|\chi_W\|_2 = \sqrt{2}$.

(c) *If* $\mathrm{End}_{\mathbb{R}}(W) \cong \mathbb{H}$, *then* $\|\chi_W\|_2 = 2$.

Proof In (a), or Case 1, $V = cW$ is irreducible and $\chi_W = \chi_V$ is a unit vector.

In (b), or Case 2, $cW = V \oplus \overline{V}$, so that $\chi_W = \chi_V + \chi_{\overline{V}}$, where χ_V and $\chi_{\overline{V}}$ are orthogonal unit vectors.

In (c), or Case 3, $cW = V \oplus \overline{V} \cong V \oplus V$, and $\chi_W = 2\chi_V$. □