Row Space and Column Space

References are to Anton–Rorres

PROBLEM: Compute everything *about the* 4×5 *matrix*

$$A=\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix} \tag{1}$$

(This includes Example 8 (p. 267) in §5.5.)

The nullspace of A *Find the dimension* (= nullity(A)) *and a basis.* In effect, *solve* the linear system $A\mathbf{x}=\mathbf{0}$. Therefore we use elementary row operations to reduce A to row echelon form (*not* uniquely, so your answer may vary)

$$R=\begin{bmatrix} \boxed{1} & -2 & 0 & 0 & 3 \\ 0 & \boxed{1} & 3 & 2 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

with the leading ones boxed. Or we can go all the way to *reduced* row echelon form

$$R'=\begin{bmatrix} \boxed{1} & 0 & 0 & -2 & 3 \\ 0 & \boxed{1} & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

which *is* unique. From (2) or (3), it is clear that x_4 and x_5 are *free* variables and may be chosen arbitrarily; we put $x_4=r$ and $x_5=s$. Then from (2), by back substitution, or directly from (3), the general solution of $A\mathbf{x}=\mathbf{0}$ can be written

$$x_1=2r-3s,\quad x_2=r,\quad x_3=-r,\quad x_4=r,\quad x_5=s$$

Thus the nullspace has dimension 2, as it needs two coordinates, and has the basis

$$\{(2,1,-1,1,0),\quad (-3,0,0,0,1)\}$$

(Here, the first vector is obtained by setting $r=1$ and $s=0$ and the second by $r=0$ and $s=1$; equivalently, we read off the coefficients of r and s in each x_j.)

The row space of A *Find the dimension* (= rank(A)) *and a basis.* By Theorem 5.5.4, the row space of A is the same as the row space of R (or R'). But by Theorem 5.5.6, we see from (2) that the first three rows of R form a basis. (None of these rows is a linear combination of later rows, and the zero row has no effect on the row space.) Thus the row space of A has dimension rank(A) = 3 and has the basis

$$\{(1,-2,0,0,3),\quad (0,1,3,2,0),\quad (0,0,1,1,0)\}$$

The column space of A *Find the dimension* (= rank(A)) *and a basis.* Write $\mathbf{u}_j$ for column j of R'. It is clear that $\mathbf{u}_1=\mathbf{e}_1$, $\mathbf{u}_2=\mathbf{e}_2$, and $\mathbf{u}_3=\mathbf{e}_3$, and that these form a basis of the column space of R'. Explicitly, we read off that $\mathbf{u}_4 = -2\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3$ and $\mathbf{u}_5 = 3\mathbf{u}_1$. The column space of R' is *not* the same as the column space of A; however, Theorem 5.5.5 allows us to conclude that the corresponding columns $\mathbf{c}_j$ of A do the same job for A. Namely, the column space of A has dimension $\operatorname{rank}(A) = 3$ and has the basis

$$\{\mathbf{c}_1 = (1,2,0,2), \quad \mathbf{c}_2 = (-2,-5,5,6), \quad \mathbf{c}_3 = (0,-3,15,18)\}$$

Further, the remaining columns of A are expressed in terms of these as

$$\mathbf{c}_4 = (0,-2,10,8) = -2\mathbf{c}_1 - \mathbf{c}_2 + \mathbf{c}_3, \quad \mathbf{c}_5 = (3,6,0,6) = 3\mathbf{c}_1$$

as is easily checked from (1).

The row space of A, revisited *Better: Select a basis from the rows of A, $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$.* Theorem 5.4.6 says that this is always possible, and sometimes, this is what we need. We just did it for the columns of A. [It is perfectly possible to develop a theory of *column* operations, but we and Anton–Rorres choose not to go this route.] *IDEA:* Consider the *transpose* of A, the 5×4 matrix

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix}$$

Elementary row operations reduce this to the row echelon form (see p. 268)

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 2 \\ 0 & \boxed{1} & -5 & -10 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

and to the reduced row echelon form

$$\begin{bmatrix} \boxed{1} & 0 & 10 & 0 \\ 0 & \boxed{1} & -5 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

Write $\mathbf{v}_j$ for column j of this matrix. This time, we have the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$, and $\mathbf{v}_3 = 10\mathbf{v}_1 - 5\mathbf{v}_2$. Again by Theorem 5.5.5 (applied to A^T), we deduce the basis

$$\{\mathbf{r}_1 = (1,-2,0,0,3), \quad \mathbf{r}_2 = (2,-5,-3,-2,6), \quad \mathbf{r}_4 = (2,6,18,8,6)\}$$

of the row space of A, and the relation (easily verified from (1))

$$\mathbf{r}_3 = (0,5,15,10,0) = 10\mathbf{r}_1 - 5\mathbf{r}_2$$

The nullspace of A^T *Find the dimension and a basis.* From (5), we see that this time there is only one free variable, x_3. The dimension is 1 and the basis consists of the single vector $(-10,5,1,0)$. Note that $1 = 4 - 3$, as in Theorem 5.6.3 (for A^T).