The Riemann Integral in Two Dimensions

See also *Step Functions in Two Dimensions*, in this series.
References are to Salas-Hille's *Calculus*, 7th Edition.

Two separate tasks: we wish to *define* and *compute* the definite integral

$$\iint_R f(x,y)\,dx\,dy$$

of a function f over the rectangle R given by $a \le x \le b$, $c \le y \le d$. The Riemann integral is based on two simple non-negotiable axioms:

$$\begin{aligned}&\text{(i) If } f \le g \text{ on } R, \text{ then } \textstyle\iint_R f(x,y)\,dx\,dy \le \iint_R g(x,y)\,dx\,dy;\\ &\text{(ii) Some functions we already know how to integrate, namely \textit{step functions}.}\end{aligned} \tag{1}$$

If s and t are step functions on R such that

$$s(x,y) \le f(x,y) \le t(x,y) \qquad \text{for all } (x,y) \in R, \tag{2}$$

axiom (i) requires

$$\iint_R s(x,y)\,dx\,dy \le \iint_R f(x,y)\,dx\,dy \le \iint_R t(x,y)\,dx\,dy, \tag{3}$$

and axiom (ii) specifies the two outer integrals. Moreover, we know that because $s \le t$, we have $\iint_R s(x,y)\,dx\,dy \le \iint_R t(x,y)\,dx\,dy$. The idea is that for favorable f, the inequality (3) is sufficient to determine the integral of f completely.

The Riemann integral

Definition 4 (cf. Defn. 16.2.3) Given a function f on R, we call f *Riemann-integrable on* R if there exists a *unique* number I such that

$$\iint_R s(x,y)\,dx\,dy \le I \le \iint_R t(x,y)\,dx\,dy \tag{5}$$

whenever s and t are step functions that satisfy (2). If this is the case, we define $\iint_R f(x,y)\,dx\,dy = I$ and call it the *Riemann integral of* f *over* R.

Note that f must be *bounded* or the definition breaks down; unless f is bounded below, s does not exist, and unless f is bounded above, t does not exist.

We have a *squeeze principle*: f is (Riemann-) integrable if and only if the difference

$$\iint_R t(x,y)\,dx\,dy - \iint_R s(x,y)\,dx\,dy = \sum_{i=1}^{m}\sum_{j=1}^{n}(t_{ij} - s_{ij})\,\text{area}(R_{ij}) \tag{6}$$

can be made arbitrarily small for suitable choices of s and t. Here we find it convenient to use (as we may) the same partition P for both s and t. Moreover, it is not necessary to check all step functions.

Lemma 7 *Suppose given a function f on R and a number I. Suppose there are step functions s and t that satisfy equations* (2) *and* (5) *and make the difference* (6) *arbitrarily small. Then f is integrable and* $\iint_R f(x,y)\,dx\,dy = I$.

Elementary properties (cf. p. 1046) These all follow directly from the corresponding statements for step functions, with the help of Lemma 7.

THEOREM 8 *Let f and g be functions on the rectangle R.*

(a) *If f is integrable on R and k is constant, then kf is integrable on R and*

$$\iint_R kf(x,y)dx\,dy = k\iint_R f(x,y)dx\,dy. \tag{9}$$

(b) *If f and g are integrable on R, so is $f+g$, and*

$$\iint_R f(x,y)+g(x,y)\,dx\,dy = \iint_R f(x,y)dx\,dy + \iint_R g(x,y)dx\,dy; \tag{10}$$

(c) *If f and g are integrable on R and $f(x,y) \le g(x,y)$ for all $(x,y) \in R$, then axiom* (1)(i) *holds,*

$$\iint_R f(x,y)dx\,dy \le \iint_R g(x,y)dx\,dy. \tag{11}$$

(d) *If we slice the rectangle R into two rectangles R' and R'' by the line $x = e$ or $y = e$, then f is integrable on R if and only if it is integrable on both R' and R'', and we have*

$$\iint_R f(x,y)dx\,dy = \iint_{R'} f(x,y)dx\,dy + \iint_{R''} f(x,y)dx\,dy. \tag{12}$$

Evaluation Let us fix y and consider the integral with respect to x,

$$F(y) = \int_a^b f(x,y)dx. \tag{13}$$

THEOREM 14 (cf. (16.4.1)) *Suppose f is integrable on R and that the integral* (13) *exists for all y (i. e. $c \le y \le d$). Then F is integrable on $[c,d]$, and*

$$\int_c^d F(y)dy = \iint_R f(x,y)dx\,dy. \tag{15}$$

Proof Choose step functions s and t such that $s \le f \le t$, and define

$$S(y) = \iint_R s(x,y)dx, \qquad T(y) = \iint_R t(x,y)dx.$$

Then $S(y) \le F(y) \le T(y)$ for all y. But S and T are again step functions, and

$$\int_c^d S(y)dy = \iint_R s \le \iint_R f \le \iint_R t = \int_c^d T(y)dy,$$

in abbreviated notation. But the difference

$$\int_c^d T(y)dy - \int_c^d S(y)dy = \iint_R (t(x,y) - s(x,y))\,dx\,dy$$

is arbitrarily small, and the squeeze principle (in one dimension) applies, to show that F is integrable and that equation (15) holds. ∎