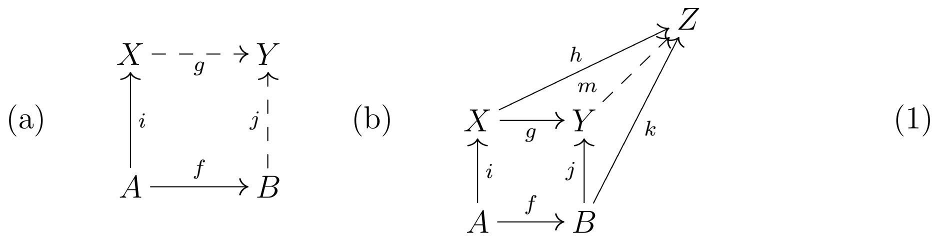
Pushouts and Adjunction Spaces

This note augments material in Hatcher, Chapter 0.

Pushouts Given maps $i: A \to X$ and $f: A \to B$, we wish to complete the commutative square (a) in a canonical way.



Definition 2 We call (1)(a) a *pushout square* if it commutes, $j \circ f = g \circ i$, and is *universal*, in the sense that given *any* space Z and maps $h: X \to Z$ and $k: B \to Z$ such that $k \circ f = h \circ i$, there exists a unique map $m: Y \to Z$ that makes diagram (1)(b) commute, $m \circ g = h$ and $m \circ j = k$. We then call Y a *pushout* of i and f.

The good news is that uniqueness of pushouts is automatic.

Proposition 3 *Given any maps i and f as in (1)(a), the pushout space Y is unique up to canonical homeomorphism.*

Proof Suppose Y', with maps $g': X \to Y'$ and $j': B \to Y'$, is another pushout. Take $Z = Y'$; we find a map $m: Y \to Y'$ such that $m \circ g = g'$ and $m \circ j = j'$. By reversing the roles of Y and Y', we find $m': Y' \to Y$ such that $m' \circ g' = g$ and $m' \circ j' = j$. Then $m' \circ m \circ g = m' \circ g' = g$, and similarly $m' \circ m \circ j = j$. Now take $Z = Y$, $h = g$, and $k = j$. We have two maps, $m' \circ m: Y \to Y$ and $\mathrm{id}_Y: Y \to Y$, that make diagram (1)(b) commute; by the uniqueness in Definition 2, $m' \circ m = \mathrm{id}_Y$. Similarly, $m \circ m' = \mathrm{id}_{Y'}$, so that m and m' are inverse homeomorphisms. □

However, existence is *not* automatic; pushouts must be constructed.

Proposition 4 *Let $i: A \to X$ and $f: A \to B$ be any maps. Then there exists a pushout Y as in Definition 2.*

Proof Let $\sim$ be the smallest equivalence relation on the topological disjoint union $X \amalg B$ that satisfies $i(a) \sim f(a)$ for all $a \in A$. [It is the intersection of all equivalence relations on $X \amalg B$ that have this property.] We take Y as the quotient space $(X \amalg B)/\sim$, with quotient map $q: X \amalg B \to Y$, and set $g = q|X$ and $j = q|B$.

We have commutativity, since for any $a \in A$, $g(i(a)) = q(i(a)) = q(f(a)) = j(f(a))$. Given h and k as in diagram (1)(b), we define $\widetilde{m}: X \amalg B \to Z$ by $\widetilde{m}|X = h$ and $\widetilde{m}|B = k$. Then for any $a \in A$, $\widetilde{m}(i(a)) = h(i(a)) = k(f(a)) = \widetilde{m}(f(a))$. It follows that $\widetilde{m}$ is constant on each equivalence class and hence factors through the map $q: X \amalg B \to Y$ to yield the desired map $m: Y \to Z$. Further, m is unique because it is required to satisfy $m \circ q = \widetilde{m}$, with $\widetilde{m}$ defined as above. □

Corollary 5 *A subset $V \subset Y$ is open (resp. closed) if and only if $g^{-1}(V)$ is open (resp. closed) in X and $j^{-1}(V)$ is open (resp. closed) in B.* □

We can stack pushout squares. The proof depends *only* on the universal property in Definition 2 and is omitted. (Try it!)

PROPOSITION 6 *Suppose given a commutative diagram*

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & B & \longrightarrow & C \end{array}$$

in which $ABXY$ is a pushout square. Then $ACXZ$ is a pushout square if and only if $BCYZ$ is a pushout square. □

Remark The third possible implication fails: if $ACXZ$ and $BCYZ$ are pushout squares, $ABXY$ need not be one. For a simple example, take $A = X = Y = C = Z$ to be a point, and B any space with more than one point.

We can also take products.

PROPOSITION 7 *If diagram* (1)(a) *is a pushout square and W is a locally compact space, then*

$$\begin{array}{ccc} X \times W & \xrightarrow{g\times\mathrm{id}} & Y \times W \\ \uparrow{\scriptstyle i\times\mathrm{id}} & & \uparrow{\scriptstyle j\times\mathrm{id}} \\ A \times W & \xrightarrow{f\times\mathrm{id}} & B \times W \end{array}$$

is another pushout square.

Proof This follows from the standard (but non-trivial) topological result that if $q\colon X \amalg B \to Y$ is a quotient map, so is $q \times \mathrm{id}_W\colon (X \amalg B) \times W \to Y \times W$. [This statement is false without some condition on W.] □

COROLLARY 8 *"The pushout of homotopies is a homotopy." Given the pushout square* (1)(a) *and homotopies $h_t\colon X \to Z$ and $k_t\colon B \to Z$ such that $h_t \circ i = k_t \circ f$ for all t, define $m_t\colon Y \to Z$ by $m_t \circ g = h_t$ and $m_t \circ j = k_t$; then m_t is a homotopy.*

Proof We take $W = I$ in Proposition 7. □

Adjunction spaces The bad news about pushouts is that Y, being a quotient space, is in general poorly behaved. Even if A, B and X are very nice spaces, Y need not even be Hausdorff. In the construction, it is far from clear what the equivalence classes in $X \coprod B$ are, or what the points of Y really are.

We shall say no more at this level of generality. From now on, we limit attention to the following special case.

PROPOSITION 9 *In the pushout square* (1)(a)*, if i is a closed embedding, so is j.*

Proof Under this hypothesis, it becomes clear what the equivalence classes in $X \amalg B$ are: they are the singletons $\{x\}$ for each $x \in X - A$, and the sets $i(f^{-1}(b)) \amalg \{b\}$ for each $b \in B$. Thus as a *set*, Y is the disjoint union of $X - A$ and B; in particular, j is injective. However, the topology on Y is not the disjoint union topology.

Recall that to prove j is a closed embedding, it is only necessary to show that $j(F)$ is closed in Y whenever F is closed in B. Because j is injective, $j^{-1}(j(F)) = F$ and $g^{-1}(j(F)) = i(f^{-1}(F))$. By Corollary 5, $j(F)$ is closed. □

Henceforth, we simplify notation by assuming that A and B actually *are* closed subspaces of X and Y, and we (usually) suppress i and j. Commutativity is simply expressed by $g|A = f$, and we have a map of pairs $g\colon (X, A) \to (Y, B)$. Informally, we obtain Y from B by *gluing* X to B along the subspace A of X as directed by the map f; we identify each point $a \in A$ with its image $f(a) \in B$.

Definition 10 When $A \subset X$ is a closed subspace, we call Y an *adjunction space* and $f\colon A \to B$ the *attaching map*. We write $Y = B \cup_f X$ (or $B \sqcup_f X$ in Hatcher).

Example If the subspaces A and B of a space X are both open or both closed, then

$$\begin{array}{ccc} A & \xrightarrow{\ \subset\ } & A \cup B \\ \uparrow{\scriptstyle\cup} & & \uparrow{\scriptstyle\cup} \\ A \cap B & \xrightarrow{\ \subset\ } & B \end{array}$$

is a pushout square. This is simply a restatement of the standard result that given a function $f\colon A \cup B \to Y$, if $f|A$ and $f|B$ are continuous, then f itself is continuous.

In general (though not always), j inherits properties from i and g from f.

Proposition 11 *Assume diagram* (1)(a) *is an adjunction square with $A \subset X$ a closed subspace. Then:*

(a) *If $A = X$, then $B = Y$;*

(b) *$g|(X - A)\colon X - A \to Y - B$ is a homeomorphism;*

(c) *If B and X are T_1 spaces, so is Y;*

(d) *If B and X are normal spaces, so is Y;*

(e) *If F is a closed subspace of X with $F \cap A = \emptyset$, then $g|F\colon F \to Y$ is a closed embedding;*

(f) *If A is a retract of X, then B is a retract of Y;*

(g) *If (X, A) satisfies the homotopy extension property, so does (Y, B);*

(h) *If A is a deformation retract of X, then B is a deformation retract of Y.*

Proof By now, (a) is trivial.

In (b), the map is obviously a continuous bijection. To see that it is an open map, take an open set V in $X - A$; then $g(V) \cap B = \emptyset$ and $g^{-1}(g(V)) = V$ show that $g(V)$ is open. Similarly for (e).

For (c), the equivalence classes in $X \amalg B$ in Proposition 9 are obviously closed.

For (d), it is convenient to understand "normal" as *not* implying T_1; then the Tietze Extension Theorem can be restated as: Y is normal if and only if any map $u\colon G \to \mathbb{R}$ from a closed subspace G of Y extends to a map $v\colon Y \to \mathbb{R}$.

Given u, because B is normal, the map $u|(B\cap G)$ extends to a map $v_B\colon B\to\mathbb{R}$. Now we put $F=g^{-1}(G)$ and work in X. The two maps $v_B\circ f\colon A\to\mathbb{R}$ and $u\circ(g|F)\colon F\to\mathbb{R}$ agree on $A\cap F$ and so define a map $A\cup F\to\mathbb{R}$. Because X is normal, this extends to a map $v_X\colon X\to\mathbb{R}$. Since $v_X|A=v_B\circ f$, we find a map $v\colon Y\to\mathbb{R}$ that satisfies $v\circ g=v_X$ and $v|B=v_B$. By construction, v extends u.

In (f), suppose $r\colon X\to A$ is a retraction, so that $r|A=\mathrm{id}_A$. We define the retraction $s\colon Y\to B$ by $s\circ g=f\circ r$ and $s|B=\mathrm{id}_B$.

In (g), suppose given a homotopy $k_t\colon B\to Z$ and a map $m_0\colon Y\to Z$ such that $m_0|B=k_0$. We have a homotopy $k_t\circ f\colon A\to Z$ and a map $m_0\circ g\colon X\to Z$ such that $(m_0\circ g)|A=m_0\circ g\circ i=m_0\circ j\circ f=k_0\circ f$; by the HEP for (X,A), there is a homotopy $h_t\colon X\to Z$ such that $h_0=m_0\circ g$ and $h_t|A=k_t\circ f$. For each t, define $m_t\colon Y\to Z$ by $m_t\circ g=h_t$ and $m_t|B=k_t$. By Corollary 8, this is the desired homotopy.

In (h), let $d_t\colon X\to X$ be a deformation retraction, so that $d_t|A=i$, $d_0=\mathrm{id}_X$, and $d_1=r$. We use the homotopy $h_t=g\circ d_t\colon X\to Y$ and the constant homotopy $k_t=j\colon B\to Y$ to construct $m_t\colon Y\to Y$, which is a homotopy by Corollary 8. We see that by uniqueness, $m_0=\mathrm{id}_Y$ and $m_1=s$, the retraction in (f). □

Examples of adjunction spaces

1. The **quotient space** X/A is obtained by taking B to be a one-point space. As a set, its points are those of $X-A$ together with one point corresponding to A.

2. The **wedge** $X\vee Y$ of two spaces X and Y with basepoints x_0 and y_0 is the quotient space $(X\amalg Y)/\{x_0,y_0\}$ obtained from the disjoint union $X\amalg Y$ by identifying the two basepoints. (It is often defined as the subspace $X\times y_0\cup x_0\times Y$ of $X\times Y$; it is easy to construct homeomorphisms between these two definitions.)

More generally, one can form the wedge $\bigvee_\alpha X_\alpha$ of any collection of based spaces (X_α,x_α) as the quotient space $(\coprod_\alpha X_\alpha)/(\coprod_\alpha x_\alpha)$.

3. The **cone** CX on X is the quotient $(X\times I)/(X\times 0)$. By (e), it contains a copy of X as the image of $X\times 1$.

4. The **suspension** SX of X is the pushout of $X\times\partial I\subset X\times I$ and the projection $X\times\partial I\to\partial I=\{0,1\}$. It contains a copy of X as the image of $X\times(1/2)$.

5. The **mapping cylinder** M_f of $f\colon A\to B$ is obtained by taking $X=A\times I$, with inclusion $A\cong A\times 1\subset A\times I$. By (h), B is a deformation retract of M_f. Also, by (e), M_f contains a copy of A as the image of $A\times 0$, as well as B.

6. The **mapping cone** C_f of $f\colon A\to B$ is obtained by taking $X=CA$, with the inclusion $A\subset CA$, or equivalently as M_f/A (with the help of Proposition 6).

7. If we take $X=D^n$, the closed unit n-disk in $\mathbb{R}^n$, and $A=S^{n-1}$, its boundary sphere, the resulting space $Y=B\cup_f D^n$, commonly written $Y=B\cup_f e^n$, is said to be obtained from B by **attaching an n-cell**, using the *attaching map* $f\colon S^{n-1}\to B$. Then $g\colon(D^n,S^{n-1})\to(Y,B)$ is called the *characteristic map* of the n-cell.

One can attach *many* n-cells by taking $X=\coprod_\alpha D^n_\alpha$ and $A=\coprod_\alpha S^{n-1}_\alpha$, using attaching maps $f_\alpha\colon S^{n-1}_\alpha\to B$, where each D^n_α is a copy of D^n, with boundary S^{n-1}_α.

8. The **smash product** $X\wedge Y$ is the quotient space $(X\times Y)/(X\vee Y)$.

9. The **join** $X*Y$ is the pushout of $X\times\partial I\times Y\subset X\times I\times Y$ and the map $X\times\partial I\times Y\to X\amalg Y$ formed from the projections $X\times 0\times Y\to X$ and $X\times 1\times Y\to Y$.