Projections and Components

Text: Salas/Hille/Etgen, *Calculus – Selected Chapters* (Wiley, 1999), Eighth Edition.

The following basic result is buried in the text as Exercise 39 to §12.4.

LEMMA 1 *Let* $\mathbf{b}$ *be a non-zero vector. Then any vector* $\mathbf{a}$ *may be decomposed uniquely as the sum*

$$\mathbf{a} = \lambda\mathbf{b} + \mathbf{c} \tag{2}$$

of a vector $\lambda\mathbf{b}$ *that is parallel to* $\mathbf{b}$ *and a vector* $\mathbf{c}$ *that is perpendicular (or orthogonal) to* $\mathbf{b}$.

Proof Suppose that equation (2) holds. We take dot products with $\mathbf{b}$. Using the distributive law, we find that

$$\mathbf{a} \bullet \mathbf{b} = \lambda\|\mathbf{b}\|^2 + \mathbf{c} \bullet \mathbf{b}.$$

Since $\mathbf{c}$ is perpendicular to $\mathbf{b}$, the last term vanishes, and the equation yields

$$\lambda = \frac{\mathbf{a} \bullet \mathbf{b}}{\|\mathbf{b}\|^2}. \tag{3}$$

So this is the only possible value for λ, and $\mathbf{c}$ must be $\mathbf{a} - \lambda\mathbf{b}$. We have uniqueness.

Conversely, these values work. The vector $\lambda\mathbf{b}$ is visibly parallel to $\mathbf{b}$, and this value of $\mathbf{c}$ is perpendicular to $\mathbf{b}$ because

$$\mathbf{c} \bullet \mathbf{b} = \mathbf{a} \bullet \mathbf{b} - \lambda\|\mathbf{b}\|^2 = 0. \quad \square$$

Remark Note that $\lambda\mathbf{b}$ will be in the *opposite* direction to $\mathbf{b}$ if λ is negative. To make the result work in all cases as stated, we have to allow $\lambda = 0$ and $\mathbf{c} = \mathbf{0}$. This puts a strain on the terminology, which is problematic from a strictly geometric point of view. If $\lambda = 0$, we have to say that the zero vector $\mathbf{0}$ is parallel to $\mathbf{b}$. If $\mathbf{c} = \mathbf{0}$, we have to say that the zero vector $\mathbf{0}$ is perpendicular to any vector.

Remark As the text points out, it is the result that is important. There is no need to introduce special notation for the vectors that appear in equation (2), and such notation is not recommended or usual outside of calculus textbooks.

This leads to a simple proof of an important inequality.

THEOREM 4 *For any vectors* $\mathbf{a}$ *and* $\mathbf{b}$*, we have the* Schwarz Inequality

$$|\mathbf{a} \bullet \mathbf{b}| \le \|\mathbf{a}\|\|\mathbf{b}\|. \tag{5}$$

Proof If $\mathbf{b} = \mathbf{0}$, both sides of equation (5) are zero.

So assume that $\mathbf{b} \ne \mathbf{0}$. In Lemma 1, we have a right-angled triangle with hypotenuse $\mathbf{a}$ and other sides $\lambda\mathbf{b}$ and $\mathbf{c}$; therefore by Pythagoras, $\|\lambda\mathbf{b}\| \le \|\mathbf{a}\|$. Now $\|\lambda\mathbf{b}\| = |\lambda|\|\mathbf{b}\|$. We plug in the value of λ from equation (3) and multiply up, to get equation (5). $\square$

The Schwarz inequality leads to an easy algebraic proof of the triangle inequality; see p. 736 in the text.

Remark All this holds without change in n dimensions.