The Natural Exponential Function

The factor $\log a$ in the derivative (see "General Exponential Functions")

$$\frac{d}{dx}(a^x) = a^x \log a \tag{1}$$

is inconvenient, and we would rather not have it.

Definition 2 We define e as the number for which $\log e = 1$. In other words,

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1\ . \tag{3}$$

In this case, eq. equation (1) simplifies to

$$\frac{d}{dx}(e^x) = e^x\ . \tag{4}$$

It is also easy to get rid of the unwanted factor by scaling. By the chain rule, $\dfrac{d}{dx}(a^{cx}) = a^{cx}(\log a)c$ for any constant c. If we choose $c = 1/\log a$ and write $a^{cx} = (a^c)^x$, we see that we can *construct* e by $e = a^c = a^{1/\log a}$ (as long as $a \neq 1$). More usefully, we can write

$$e^{\log a} = (a^{1/\log a})^{\log a} = a^1 = a, \tag{5}$$

and hence

$$a^x = (e^{\log a})^x = e^{x\log a}. \tag{6}$$

More differentiation When we differentiate equation (5) in the form $e^{\log x} = x$, we obtain (assuming that $\log x$ *is* differentiable) $e^{\log x}\dfrac{d}{dx}(\log x) = 1$, which we rearrange as

$$\frac{d}{dx}(\log x) = \frac{1}{e^{\log x}} = \frac{1}{x}\ . \tag{7}$$

Now that we have equation (6), we can vary a instead of (or as well as) x. For example, for any real constant c we have

$$\frac{d}{dx}(x^c) = \frac{d}{dx}(e^{c\log x}) = e^{c\log x}\frac{d}{dx}(c\log x) = x^c\frac{c}{x} = cx^{c-1}, \tag{8}$$

which we previously had only for rational c.

Future directions We have deferred the proofs of existence of a^x and $\log a$ and the differentiability of $\log x$, nothing else. It is rather hard to prove these directly with what we have. For the sake of efficiency, we observe from equation (6) that it is only necessary to develop the functions e^x and $\log x$, which by equation (5) are inverse to each other. This is what we shall do in due course.