Methods of Integration

References are to Thomas & Finney, 8th edition.

Integration The *definition* of the indefinite integral is

$$\int du = u + C \qquad \text{where } C \text{ is an arbitrary constant} \tag{1}$$

for any variable u. The workhorse of integration is the method of *substitution* (or change of variable); see the flowchart [p. 517]. Integration is *linear:*

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx \tag{2}$$

and

$$\int kf(x)dx = k \int f(x)dx \qquad \text{where } k \text{ is constant.} \tag{3}$$

Polynomials For polynomials we use

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \qquad \text{provided } n \neq -1 \tag{4}$$

where n may also be fractional or negative. The special case $n = -1$ is handled by

$$\int \frac{du}{u} = \log u + C \qquad \text{or } \log(-u) \text{ if } u \text{ is negative.} \tag{5}$$

Rational functions, I [See §7.5, esp. pp. 510-511.] Rational functions $\dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, can always be integrated.

If this fraction is *improper*, i. e. $\deg p(x) \geq \deg q(x)$, we must first use *polynomial division*

$$p(x) = q(x)s(x) + r(x),$$

where $\deg r(x) < \deg q(x)$, to write it in terms of a proper fraction as

$$\frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)}$$

We assume for now that $q(x)$ is a product of *linear* factors. If the a_i are all distinct, any proper fraction can be decomposed uniquely as

$$\frac{p(x)}{(x-a_1)(x-a_2)\dots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \dots + \frac{A_n}{x-a_n} \tag{6}$$

for suitable constants A_i. This is the method of *partial fractions.* To find the A_i, clear the denominators and equate coefficients of powers of x, or choose various values for x, to obtain enough equations to determine the A_i. (Here, putting $x = a_i$ is especially useful.) Then equation (6) is easily integrated by using equations (2) and (5) with the substitutions $u = x - a_i$ and $du = dx$.

If the a_i are not all distinct, equation (6) is clearly inappropriate because the common denominator is wrong. In general, if $q(x)$ has the repeated linear factor $(x-a)^m$, we must replace the m identical terms $\dfrac{A}{x-a}$ in equation (6) by

$$\frac{B_1}{x-a} + \frac{B_2}{(x-a)^2} + \ldots + \frac{B_m}{(x-a)^m}\ . \tag{7}$$

This is easily integrated by equations (5) and (4).

Exponential functions These are handled by

$$\int e^u du = e^u + C \tag{8}$$

Trigonometric functions The six trigonometric functions of x may be expressed in terms of $\cos x$ and $\sin x$, so that the basic trigonometric polynomial integral takes the form $\int \sin^m x \cos^n x dx$. We can also allow m or n to be negative.

Case m odd We put $u = \cos x$ and $du = -\sin x dx$ and use $\sin^2 x = 1 - u^2$ on the remaining even powers of $\sin x$, to get a rational function of u.

Case n odd We put $u = \sin x$ and $du = \cos x dx$ and use $\cos^2 x = 1 - u^2$ on the remaining even powers of $\cos x$, to get a rational function of u.

Example One important and useful application is the integral

$$\int \sec x dx = \int \frac{\cos x dx}{\cos^2 x} = \int \frac{du}{1-u^2} = \ldots = \log(\sec x + \tan x) + C \tag{9}$$

Here, we use partial fractions to write

$$\frac{1}{1-u^2} = \frac{1}{(1+u)(1-u)} = \frac{1/2}{1+u} + \frac{1/2}{1-u}\ .$$

By equation (5), this integrates to give

$$\frac{1}{2}\log(1+u) - \frac{1}{2}\log(1-u) = \frac{1}{2}\log\frac{1+\sin x}{1-\sin x}$$

To clean this up, we write

$$\begin{aligned}\frac{1+\sin x}{1-\sin x} &= \frac{(1+\sin x)(1+\sin x)}{(1-\sin x)(1+\sin x)} = \frac{(1+\sin x)^2}{1-\sin^2 x}\\ &= \frac{(1+\sin x)^2}{\cos^2 x} = \left(\frac{1+\sin x}{\cos x}\right)^2 = (\sec x + \tan x)^2\end{aligned}$$

and use $\log z^2 = 2\log z$.

Similarly, or by putting $y = \pi/2 - x$ in equation (9), we have

$$\int \csc y dy = -\log(\csc y + \cot y) + C \tag{10}$$

Case m and n even In this case we can use the double angle formulae

$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

to obtain an integral involving only $\cos 2x$. Repeat if necessary.

If n is negative, the substitution $u = \tan x$, $du = \sec^2 x dx$ can be useful.

For integrals of the form $\int \sin mx \sin nx dx$ etc., see p. 497.

Rational functions, II Not all polynomials have linear factors. However, we do have the fundamental theorem of real algebra:

THEOREM 11 *Every polynomial $x^n + \ldots$ in x factors uniquely up to order as a product of:*

(i) *linear factors of the form $x - a$;*

(ii) *quadratic factors of the form $x^2 + ax + b$ that have no real root.*

When $q(x)$ has a quadratic factor $x^2 + ax + b$, the appropriate term of the partial fraction decomposition must be taken as

$$\frac{Ax + B}{x^2 + ax + b} \tag{12}$$

in order to provide enough indeterminates. For a repeated quadratic factor $(x^2+ax+b)^m$, we need instead

$$\frac{A_1x + B_1}{x^2 + ax + b} + \frac{A_2x + B_2}{(x^2 + ax + b)^2} + \ldots + \frac{A_mx + B_m}{(x^2 + ax + b)^m} \tag{13}$$

Quadratic denominators (See §7.4.) First complete the square, if necessary,

$$x^2 + ax + b = (x + c)^2 + f^2$$

where $c = a/2$ and $f = \sqrt{b - a^2/4}$, and make the linear substitution $u = x + c$ and $du = dx$.

We break up equation (12) into two terms. In the first, the substitution $u = f \tan\theta$, $du = f \sec^2\theta d\theta$ gives

$$\int \frac{du}{u^2 + f^2} = \int \frac{f \sec^2\theta d\theta}{f^2 \sec^2\theta} = \frac{1}{f}\theta + C = \frac{1}{f}\tan^{-1}\frac{u}{f} + C \tag{14}$$

In the second, we simply put $s = u^2 + f^2$, $ds = 2udu$, to get

$$\int \frac{udu}{u^2 + f^2} = \int \frac{ds}{2s} = \frac{1}{2}\log s + C = \frac{1}{2}\log(u^2 + f^2) + C \tag{15}$$

However, the substitution $u = f\tan\theta$ works here too, somewhat less efficiently.

The same substitutions also handle the integrals

$$\int \frac{du}{(u^2 + f^2)^m} \qquad \int \frac{udu}{(u^2 + f^2)^m}$$

with repeated quadratic factors.