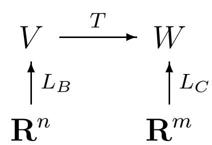
Linear Transformations and Matrices

References are to Anton–Rorres, 7th Edition
See also "Coordinate Vectors", in this series

We wish to describe a general linear transformation $T: V \to W$ in terms of coordinate systems on V and W. Suppose we are given bases $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of V and $C = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m\}$ of W. We use the resulting linear isomorphisms $L_B: \mathbf{R}^n \to V$ and $L_C: \mathbf{R}^m \to W$ to convert T to a linear transformation $\mathbf{R}^n \to \mathbf{R}^m$, which we know how to handle. Diagramatically, we have



Definition 1 The *matrix of* $T: V \to W$ *with respect to the bases* B *and* C is the matrix A of the composite linear transformation $L_C^{-1} \circ T \circ L_B: \mathbf{R}^n \to \mathbf{R}^m$. We write it $[T]_{C,B}$ (the order here may seem perverse; but see below). If $W = V$ and $C = B$, we write simply $[T]_B$.

Remark If $W = V$ but we choose *different* bases B and C of V, the matrix $[I]_{B,C}$ of the identity linear transformation $I: V \to V$ is the transition matrix P from C to B. This is Theorem 8.5.1. To see this, just compare the definitions (and watch the order of B and C!).

To make A explicit, we need to compute the effect of this linear transformation on $\mathbf{e}_i$. First, $L_B(\mathbf{e}_i) = \mathbf{b}_i$. Then we apply T, to get $T(\mathbf{b}_i)$. Finally, $L_C^{-1}(T(\mathbf{b}_i))$ is by definition the coordinate vector $[T(\mathbf{b}_i)]_C$ of $T(\mathbf{b}_i)$ with respect to C. Thus the matrix A of T is

$$A = \Big[[T(\mathbf{b}_1)]_C \Big| [T(\mathbf{b}_2)]_C \Big| \ldots \Big| [T(\mathbf{b}_n)]_C\Big] \tag{2}$$

It does what we expect on coordinate vectors.

Theorem 3 *If A is the matrix of $T: V \to W$ with respect to bases B and C, then*

$$A[\mathbf{v}]_B = [T(\mathbf{v})]_C \qquad \textit{for all } \mathbf{v} \in V. \tag{4}$$

Proof This is simply the translation of the statement

$$(L_C^{-1} \circ T \circ L_B)(L_B^{-1}(\mathbf{v})) = L_C^{-1}(T(\mathbf{v})). \quad \square$$

Composition Matrices behave correctly under composition of linear transformations.

Theorem 5 (=Thm. 8.4.2) *Let V, W, and X be vector spaces with bases B, C and D respectively. If A is the matrix of the linear transformation $T: V \to W$ with respect to B and C, and A' is the matrix of the linear transformation $U: W \to X$, then the matrix of the composite linear transformation $U \circ T: V \to X$ is the product matrix $A'A$. In symbols, $[U \circ T]_{D,B} = [U]_{D,C} \circ [T]_{C,B}$.*

Proof This is the translation of the statement

$$L_D^{-1} \circ (U \circ T) \circ L_B = (L_D^{-1} \circ U \circ L_C) \circ (L_C^{-1} \circ T \circ L_B). \quad \square$$

The relevant diagram is

$$\begin{array}{ccccc} V & \xrightarrow{T} & W & \xrightarrow{U} & X \\ \uparrow L_B & & \uparrow L_C & & \uparrow L_D \\ \mathbf{R}^n & & \mathbf{R}^m & & \mathbf{R}^p \end{array}$$

Similarity We need to know the effect of a change of basis on the matrix of a linear operator $T: V \to V$.

THEOREM 6 (=Thm. 8.5.2) *Let A be the matrix of the linear operator $T: V \to V$ with respect to the basis B of V. Let C be another basis of V, with P the transition matrix from C to B. Then the matrix of T with respect to C is $P^{-1}AP$. In symbols,*

$$[T]_C = P^{-1}[T]_B P. \tag{7}$$

Proof This translates the obvious statement

$$L_C^{-1} \circ T \circ L_C = (L_C^{-1} \circ L_B) \circ (L_B^{-1} \circ T \circ L_B) \circ (L_B^{-1} \circ L_C) \quad \square$$

Change of basis The general case is no harder.

THEOREM 8 *Let B and B' be bases of V, and P the transition matrix from B' to B. Let C and C' be bases of W, and Q the transition matrix from C' to C. If the matrix of $T: V \to W$ with respect to B and C is A, then the matrix of T with respect to B' and C' is $Q^{-1}AP$.*

Proof This translates the statement

$$L_{C'}^{-1} \circ T \circ L_{B'} = (L_{C'}^{-1} \circ L_C) \circ (L_C^{-1} \circ T \circ L_B) \circ (L_B^{-1} \circ L_{B'}). \quad \square$$