Linear System Example: Distinct Real Roots

We solve Example 1 of §56 in [Simmons, Second edition], on p. 429,

$$\begin{cases} \dfrac{dx}{dt} &= x + y \\ \dfrac{dy}{dt} &= 4x - 2y \end{cases}$$

by an alternate method. We apply the Laplace transform with the generic initial conditions $x(0) = k_1$ and $y(0) = k_2$ to get

$$\begin{cases} pX - k_1 &= X + Y \\ pY - k_2 &= 4X - 2Y \end{cases} \quad \text{or} \quad \begin{cases} (1-p)X + Y &= -k_1 \\ 4X + (-2-p)Y &= -k_2. \end{cases}$$

The resemblance to (9) is not accidental. We solve these simultaneous linear equations for X and Y by elimination,

$$\begin{cases} [(-2-p)(1-p) - 1\cdot 4]X &= -(-2-p)k_1 + k_2 &= pk_1 + 2k_1 + k_2 \\ [(1-p)(-2-p) - 4\cdot 1]Y &= -(1-p)k_2 + 4k_1 &= pk_2 + 4k_1 - k_2. \end{cases}$$

The expressions in [ ] are both

$$(-2-p)(1-p) - 4 = p^2 + p - 6 = (p+3)(p-2).$$

We therefore divide out by this and use (easy) partial fractions,

$$\begin{cases} X &= \dfrac{pk_1 + 2k_1 + k_2}{(p+3)(p-2)} &= \dfrac{4k_1 + k_2}{5(p-2)} + \dfrac{k_1 - k_2}{5(p+3)} \\ Y &= \dfrac{pk_2 + 4k_1 - k_2}{(p+3)(p-2)} &= \dfrac{4k_1 + k_2}{5(p-2)} + \dfrac{-4k_1 + 4k_2}{5(p+3)} \end{cases}$$

Finally, we apply the inverse Laplace transform and find

$$\begin{cases} x &= \dfrac{4k_1 + k_2}{5}e^{2t} + \dfrac{k_1 - k_2}{5}e^{-3t} \\ y &= \dfrac{4k_1 + k_2}{5}e^{2t} - \dfrac{4(k_1 - k_2)}{5}e^{-3t}. \end{cases}$$

This agrees with (12) in the book, if we take $c_1 = \dfrac{k_1 - k_2}{5}$ and $c_2 = \dfrac{4k_1 + k_2}{5}$.

Compare this treatment with the traditional one in the book to decide which is shorter or easier or more efficient. Draw your own conclusions.