Linear System Example: Complex Roots

We solve Problem 1(h) of §56 in [Simmons, Second edition], on p. 433,

$$\begin{cases} \dfrac{dx}{dt} &= x - 2y \\[2mm] \dfrac{dy}{dt} &= 4x + 5y \end{cases}$$

by two different methods. (No similar example is worked out in full in the book.) Again, compare the two treatments to decide which is shorter or easier or more efficient. Draw your own conclusions.

(a) By the traditional method   We try $x = Ae^{mt}$ and $y = Be^{mt}$, which works (as in (3)) provided A, B, and m satisfy the *eigenvalue problem*

$$\begin{cases} mA &= A - 2B \\ mB &= 4A + 5B \end{cases} \quad \text{or} \quad \begin{cases} (1-m)A - 2B &= 0 \\ 4A + (5-m)B &= 0 \ . \end{cases}$$

This has solutions for A and B (other than $A = B = 0$) provided that

$$0 = (1 - m)(5 - m) + 8 = m^2 - 6m + 13 = (m - 3)^2 + 2^2 \ .$$

The eigenvalues are therefore $m = 3 \pm 2i$.

We choose $m = 3 + 2i$. Then the eigenvector (A, B) has to satisfy

$$\begin{cases} (-2 - 2i)A - 2B &= 0 \\ 4A + (2 - 2i)B &= 0 \end{cases}$$

The solution $A = 1$, $B = -1 - i$ is as simple as any, and yields the *complex* solution

$$\begin{cases} x &= e^{3t+2it} &= e^{3t}(\cos 2t + i \sin 2t) \\ y &= -(1+i)e^{3t+2it} &= e^{3t}(-\cos 2t + \sin 2t - i \cos 2t - i \sin 2t) \ . \end{cases}$$

We want real solutions. The key point is that because the given system is a *real* system, the real and imaginary parts of (x, y) will automatically be solutions too. This observation yields the two real solutions

$$\begin{cases} x &= e^{3t} \cos 2t \\ y &= e^{3t}(-\cos 2t + \sin 2t) \end{cases} \quad \text{and} \quad \begin{cases} x &= e^{3t} \sin 2t \\ y &= -e^{3t}(\cos 2t + \sin 2t) \end{cases}$$

which are clearly independent and lead to the real general solution

$$\begin{cases} x &= e^{3t}(c_1 \cos 2t + c_2 \sin 2t) \\ y &= e^{3t}[-(c_1 + c_2) \cos 2t + (c_1 - c_2) \sin 2t] \ , \end{cases}$$

which happens to be exactly the answer in the book.

We could easily have chosen a different eigenvector (A, B), say $A = -(1 - i)$ and $B = 2$, which yields the complex solution

$$\begin{cases} x &=& -(1 - i)e^{3t}(\cos 2t + i \sin 2t) = e^{3t}[- \cos 2t - \sin 2t + i(\cos 2t - \sin 2t)] \\ y &=& 2e^{3t}(\cos 2t + i \sin 2t) \end{cases}$$

and corresponding real general solution

$$\begin{cases} x &=& e^{3t}[(b_2 - b_1) \cos 2t - (b_1 + b_2) \sin 2t] \\ y &=& e^{3t}(2b_1 \cos 2t + 2b_2 \sin 2t) \ . \end{cases}$$

Despite appearances, this is the same as before, with $b_1 = -\dfrac{c_1 + c_2}{2}$ and $b_2 = \dfrac{c_1 - c_2}{2}$.

There is never any need to consider the other root, $m = 3 - 2i$, which cannot and does not give anything new.

(b) By Laplace transforms   We take the Laplace transform of the system, with the generic initial conditions $x(0) = k_1$ and $y(0) = k_2$, to get

$$\begin{cases} pX - k_1 &=& X - 2Y \\ pY - k_2 &=& 4X + 5Y \end{cases} \quad \text{or} \quad \begin{cases} (1 - p)X - 2Y &=& -k_1 \\ 4X + (5 - p)Y &=& -k_2 \ . \end{cases}$$

We solve these simultaneous linear equations for X and Y by elimination,

$$\begin{cases} [(5 - p)(1 - p) + 2 \cdot 4]X &=& -(5 - p)k_1 - 2k_2 &=& (p - 5)k_1 - 2k_2 \\ [(1 - p)(5 - p) - 4 \cdot (-2)]Y &=& -(1 - p)k_2 + 4k_1 &=& 4k_1 + (p - 1)k_2 \ . \end{cases}$$

The expressions in [ ] are both

$$(5 - p)(1 - p) + 8 = p^2 - 6p + 13 = (p - 3)^2 + 2^2 \ .$$

We therefore divide out by this and express everything in terms of $p - 3$, with an eye towards using the shift formula for Laplace transforms,

$$\begin{cases} X &=& \dfrac{(p - 5)k_1 - 2k_2}{(p - 3)^2 + 2^2} &=& \dfrac{(p - 3)k_1}{(p - 3)^2 + 2^2} - \dfrac{2k_1 + 2k_2}{(p - 3)^2 + 2^2} \\ Y &=& \dfrac{4k_1 + (p - 1)k_2}{(p - 3)^2 + 2^2} &=& \dfrac{(p - 3)k_2}{(p - 3)^2 + 2^2} + \dfrac{4k_1 + 2k_2}{(p - 3)^2 + 2^2} \end{cases}$$

Finally, we apply the inverse Laplace transform and find

$$\begin{cases} x &=& e^{3t}[k_1 \cos 2t - (k_1 + k_2) \sin 2t] \\ y &=& e^{3t}[k_2 \cos 2t + (2k_1 + k_2) \sin 2t] \ . \end{cases}$$

This agrees with the answer in the book, if we write $k_1 = c_1$ and $k_2 = -(c_1 + c_2)$, as is easily seen by checking the initial conditions. Note that in this method, there is no complex anything.