## Inverse Linear Substitutions and Matrices

We generalize Example 3 (on page 5 of Anton-Rorres) by solving the linear system

$$
\left\{\begin{aligned}
& x+y+2 z=u \\
& \\
& 2 x+4 y-3 z= \\
& 3 x+6 y-5 z= \\
& \\
& 3 x \\
& \\
&(1) \\
&(2)
\end{aligned} \quad \text { with matrix } A=\left[\begin{array}{rrr}
1 & 1 & 2 \\
2 & 4 & -3 \\
3 & 6 & -5
\end{array}\right]\right.
$$

for $x, y$, and $z$ in terms of $u, v$, and $w$. In the book, $u=9, v=1$, and $w=0$. The point is that we can solve the more general problem with very little extra work, by following exactly the same steps as before (annotated symbolically).

$$
\begin{array}{rlrl|l}
2 y-7 z & =-2 u \quad+v & (4) & =(2)-2(1) \\
3 y-11 z & =-3 u & +w & (5) & =(3)-3(1) \\
y-\frac{7}{2} z & =-u+\frac{1}{2} v & (6) & =\frac{1}{2}(4) \\
-\frac{1}{2} z & = & -\frac{3}{2} v & +w & (7) \\
z & =3 v & =(5)-3(6) \\
x+\frac{11}{2} z & =2 u & (8) & =-2(7) \\
x & & -\frac{1}{2} v & (9) & =(1)-(6) \\
x & 2 u-17 v+11 w & (10) & =(9)-\frac{11}{2}(8) \\
y & =-u+11 v & -7 w & (11) & =(6)+\frac{7}{2}(8)
\end{array}
$$

Thus the desired solution consists of equations (10), (11), and (8),

$$
\left\{\begin{array}{rrrr}
x & = & 2 u & -17 v \\
y & +11 w \\
y & -u & +11 v & -7 w \\
z & 3 v & -2 w
\end{array} \quad \text { with matrix } B=\left[\begin{array}{rrr}
2 & -17 & 11 \\
-1 & 11 & -7 \\
0 & 3 & -2
\end{array}\right]\right.
$$

This is the inverse linear substitution, whose matrix $B$ is the inverse matrix $A^{-1}$ to $A$. In matrix notation, we solved the equation $A \mathbf{x}=\mathbf{u}$ and obtained the solution $\mathbf{x}=B \mathbf{u}$, where we write

$$
\mathbf{x}=\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right], \quad \mathbf{u}=\left[\begin{array}{c}
u \\
v \\
w
\end{array}\right]
$$

Computationally, we in effect applied row reduction to the $3 \times 6$ partitioned matrix $[A \mid I]$ to produce the reduced row echelon matrix $[I \mid B]$,

$$
\left[\begin{array}{rrr|rrr}
1 & 1 & 2 & 1 & 0 & 0 \\
2 & 4 & -3 & 0 & 1 & 0 \\
3 & 6 & -5 & 0 & 0 & 1
\end{array}\right] \longrightarrow\left[\begin{array}{lll|rrr}
1 & 0 & 0 & 2 & -17 & 11 \\
0 & 1 & 0 & -1 & 11 & -7 \\
0 & 0 & 1 & 0 & 3 & -2
\end{array}\right]
$$

(If it had failed to reduce to this form, the message would have been that $A$ was not invertible.)

