Inverse Linear Substitutions and Matrices

We generalize Example 3 (on page 5 of Anton–Rorres) by solving the linear system

$$\begin{cases} x + y + 2z = u & (1) \\ 2x + 4y - 3z = v & (2) \\ 3x + 6y - 5z = w & (3) \end{cases} \text{ with matrix } A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$$

for x, y, and z in terms of u, v, and w. In the book, u = 9, v = 1, and w = 0. The point is that we can solve the more general problem with very little extra work, by following exactly the same steps as before (annotated symbolically).

Thus the desired solution consists of equations (10), (11), and (8),

$$\begin{cases} x = 2u - 17v + 11w \\ y = -u + 11v - 7w \\ z = 3v - 2w \end{cases} \text{ with matrix } B = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$$

This is the *inverse* linear substitution, whose matrix B is the *inverse matrix* A^{-1} to A. In matrix notation, we solved the equation $A\mathbf{x} = \mathbf{u}$ and obtained the solution $\mathbf{x} = B\mathbf{u}$, where we write

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Computationally, we in effect applied row reduction to the 3×6 partitioned matrix [A|I] to produce the reduced row echelon matrix [I|B],

[1	1	2	1	0	0]		[1]	0	0	2	-17	11]
2	4	-3	0	1	0	\longrightarrow						
$\left[\begin{array}{c}1\\2\\3\end{array}\right]$	6	-5	0	0	1		0	0	1	0	3	-2

(If it had failed to reduce to this form, the message would have been that A was not invertible.)

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