Local Maxima and Minima

References are to Salas/Hille/Etgen's *Calculus*, 8th Edition

We study the behavior of the scalar-valued function $f(\mathbf{r})$ of the 2-dimensional vector variable $\mathbf{r}$ near a *stationary* point $\mathbf{r}_0$ (one where $\nabla f(\mathbf{r}_0) = \mathbf{0}$). We wish to determine whether f has a local maximum or minimum at $\mathbf{r}_0$. We assume throughout that f has class C^2 (all second-order partial derivatives of f exist and are continuous).

There is one situation that occurs naturally in n dimensions, for $n \geq 2$.

DEFINITION 1 The point $\mathbf{r}_0$ is a *saddle point* of f if it is stationary ($\nabla f(\mathbf{r}_0) = \mathbf{0}$), but is neither a local maximum nor a local minimum of f. (For a good picture, see p. 916 or p. 834; you can fill in the horse.)

The second derivative We reduce to a one-dimensional problem by restricting attention to the values of f along a line through $\mathbf{r}_0$. We choose any vector $\mathbf{h} = h_1\mathbf{i}+h_2\mathbf{j}$ and set $g(t) = f(\mathbf{r}_0 + t\mathbf{h})$. Then by the chain rule,

$$g'(t) = \nabla f(\mathbf{r}_0 + t\mathbf{h}) \bullet \mathbf{h} = f_x(\mathbf{r}_0 + t\mathbf{h})\, h_1 + f_y(\mathbf{r}_0 + t\mathbf{h})\, h_2.$$

In particular, $g'(0) = 0$, and Taylor's Theorem simplifies to

$$g(t) = f(\mathbf{r}_0 + t\mathbf{h}) = g(0) + \frac{t^2}{2} g''(\theta t) = f(\mathbf{r}_0) + \frac{t^2}{2} g''(\theta t), \tag{2}$$

for some θ (depending on t), where $0 < \theta < 1$. It is now clear by continuity that:

(a) If $g''(0) > 0$, then g has a strict local minimum at $t = 0$. In other words, $g(t) > g(0)$ for all small $t \neq 0$. So $g(t)$ does *not* have a local *maximum* at $t = 0$, and $f(\mathbf{r})$ does not have a local maximum at $\mathbf{r}_0$.

(b) If $g''(0) < 0$, then g has a strict local maximum at $t = 0$. In other words, $g(t) < g(0)$ for all small $t \neq 0$. So $g(t)$ does not have a local minimum at $t = 0$, and $f(\mathbf{r})$ does not have a local minimum at $\mathbf{r}_0$. (3)

(c) If $g''(0) = 0$, we have *no information* about $g''(\theta t)$ or $g(t)$.

We compute that

$$g''(0) = Q(\mathbf{h}) = Ah_1^2 + 2Bh_1h_2 + Ch_2^2, \tag{4}$$

a *quadratic form* in $\mathbf{h}$, which we abbreviate to $Q(\mathbf{h})$, with coefficients

$$A = f_{xx}(\mathbf{r}_0), \qquad B = f_{xy}(\mathbf{r}_0), \qquad C = f_{yy}(\mathbf{r}_0).$$

Classification There are *six* types of quadratic form $Q(\mathbf{h})$:

(i) *positive-definite*: $Q(\mathbf{h}) > 0$ for all $\mathbf{h} \neq \mathbf{0}$;

(ii) *positive-semidefinite*: $Q(\mathbf{h})$ takes both positive and zero values for $\mathbf{h} \neq \mathbf{0}$;

(iii) *negative-definite*: $Q(\mathbf{h}) < 0$ for all $\mathbf{h} \neq \mathbf{0}$;

(iv) *negative-semidefinite*: $Q(\mathbf{h})$ takes both negative and zero values for $\mathbf{h} \neq \mathbf{0}$;

(v) *indefinite*: $Q(\mathbf{h})$ takes both positive and negative values (and hence also zero values) for $\mathbf{h} \neq \mathbf{0}$;

(vi) *zero*: $Q(\mathbf{h}) = 0$ for all $\mathbf{h}$.

Except for (vi), the type of a given quadratic form $Q(\mathbf{h})$ is *not* obvious in general.

Case 1: $A \neq 0$. We *complete the square*, by rewriting equation (4) as

$$Q(\mathbf{h}) = A\left(h_1 + \frac{B}{A}h_2\right)^2 + \left(C - \frac{B^2}{A}\right)h_2^2 = A\left(h_1 + \frac{B}{A}h_2\right)^2 - \frac{D}{A}h_2^2, \tag{5}$$

where we introduce the *discriminant* $D = B^2 - AC$ of $Q(\mathbf{h})$. [*WARNING:* Many books (with good reason) change the sign and consider $AC - B^2$ rather than $B^2 - AC$.] Since the expressions h_2 and $h_1 + (B/A)h_2$ can be chosen independently, all we have to do is check the signs of the coefficients A and $-D/A$.

Case 2: $A = 0$. Here, $D = B^2$, and we write

$$Q(\mathbf{h}) = (2Bh_1 + Ch_2)h_2. \tag{6}$$

THEOREM 7 *The behavior of f near $\mathbf{r}_0$ is given by the following table:*

Type of $Q(\mathbf{h})$	Conditions	Local minimum?	Local maximum?
Positive-definite	$D < 0,\ A > 0$	yes, strict	no
Positive-semidef.	$D = 0,\ (A > 0$ or $C > 0)$	maybe	no
Negative-definite	$D < 0,\ A < 0$	no	yes, strict
Negative-semidef.	$D = 0,\ (A < 0$ or $C < 0)$	no	maybe
Indefinite	$D > 0$	no, saddle point	no, saddle point
Zero	$A = B = C = 0$	maybe	maybe

This includes everything in Theorem 15.5.3, and a little more. Simple examples show that all statements are best possible. These results remain valid in n dimensions (with the second column greatly generalized, in ways that are not obvious).

Proof We read off the conditions from equations (5) and (6). The "no" statements follow directly from (3) and equation (5) or (6). If there exists $\mathbf{h}$ such that $Q(\mathbf{h}) > 0$, then f does not have a local maximum; if there exists $\mathbf{h}$ such that $Q(\mathbf{h}) < 0$, then f does not have a local minimum.

For the positive-definite case, we use equation (2) to write

$$f(\mathbf{r}_0 + \mathbf{h}) = g(1) = f(\mathbf{r}_0) + \frac{1}{2}Q(\mathbf{h}),$$

where $Q(\mathbf{h})$ is given by equation (5), except that we must now evaluate A, B, C and D at $\mathbf{r}_0 + \theta\mathbf{h}$ rather than $\mathbf{r}_0$. Since these functions are all continuous, we still have $D < 0$ and $A > 0$ for all small $\mathbf{h}$, and it is clear that $Q(\mathbf{h}) > 0$ for all small $\mathbf{h} \neq \mathbf{0}$.

The negative-definite case is entirely similar. Alternatively, we can simply consider $-f$ instead of f. □