Linear Substitutions and Matrix Multiplication

This note interprets matrix multiplication and related concepts in terms of the composition of linear substitutions.

Composition and multiplication We start from the *linear substitution* (cf. Example 3 on page 5 of Anton–Rorres)

$$\left\{\begin{array}{rcrrr} u &=& x & +y & +2z \\ v &=& 2x & +4y & -3z \\ w &=& 3x & +6y & -5z \end{array}\right. \qquad \text{with matrix} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \tag{1}$$

(Linear substitutions are usually not allowed to have constant terms.) Then we substitute in turn linear expressions for each of x, y, and z:

$$\left\{\begin{array}{rcrr} x &=& s & +2t \\ y &=& 3s & -t \\ z &=& -s & +t \end{array}\right. \qquad \text{with matrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ -1 & 1 \end{bmatrix} \tag{2}$$

We see that u, v, and w then become linear expressions in s and t,

$$\left\{\begin{array}{rlll} u &= x+y+2z &= s+2t+3s-t-2s+2t &= 2s+3t \\ v &= 2x+4y-3z &= 2s+4t+12s-4t+3s-3t &= 17s-3t \\ w &= 3x+6y-5z &= 3s+6t+18s-6t+5s-5t &= 26s-5t \end{array}\right.$$

This is the *composite* linear substitution of the linear substitutions represented by equations (1) and (2). We observe that its matrix D is just the *product* matrix AB,

$$D = AB = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 17 & -3 \\ 26 & -5 \end{bmatrix}$$

In other words, in terms of the vectors

$$\mathbf{s} = \begin{bmatrix} s \\ t \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

we have

$$\mathbf{u} = A\mathbf{x} = A(B\mathbf{s}) = (AB)\mathbf{s} = D\mathbf{s} \tag{3}$$

Associativity Suppose we introduce another linear substitution $\mathbf{s} = C\mathbf{p}$, where we find that there will be no need to make $\mathbf{p}$ explicit. Then we can express $\mathbf{x}$ in terms of $\mathbf{p}$ by the linear substitution

$$\mathbf{x} = B\mathbf{s} = B(C\mathbf{p}) = E\mathbf{p}, \tag{4}$$

where $E = BC$ denotes another product matrix.

We now have *two* ways to express $\mathbf{u}$ in terms of $\mathbf{p}$. We can use equation (3) to go by way of $\mathbf{s}$,

$$\mathbf{u} = D\mathbf{s} = D(C\mathbf{p}) = (DC)\mathbf{p}$$

Or we can use equations (1) and (4) to go by way of $\mathbf{x}$,

$$\mathbf{u} = A\mathbf{x} = A(E\mathbf{p}) = (AE)\mathbf{p}$$

We therefore conclude, without doing any real work, that $AE = DC$, that is, $A(BC) = (AB)C$, which is the *associative law*. Alternatively, we could express this diagrammatically as

$$\mathbf{p} \xrightarrow{C} \mathbf{s} \xrightarrow{B} \mathbf{x} \xrightarrow{A} \mathbf{u}$$

which indicates that $(AB)C$ and $A(BC)$ are really the same. We may simply write $\mathbf{u} = ABC\mathbf{p}$, with no parentheses.

The identity The *identity* linear substitution

$$\begin{cases} x = x \\ y = \quad y \\ z = \qquad z \end{cases} \qquad \text{whose matrix is} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

clearly preserves everything. Its matrix I (in this case I_3) is therefore called the *identity* matrix. Generally, there is an $n \times n$ identity matrix I_n for each n. (But it has to be square.) From this point of view, it is obvious that $I_m F = F = F I_n$ for any $m \times n$ matrix F.

The zero In contrast, the *zero* linear substitution

$$\begin{cases} y_1 = 0 \\ y_2 = 0 \\ y_3 = 0 \\ y_4 = 0 \end{cases}$$

destroys everything in sight by setting all variables zero. Its matrix is the *zero matrix*, with all entries 0. The zero matrix comes in all sizes $m \times n$. Applying other linear substitutions before or after the zero linear substitution will still leave all variables set to 0. So in matrix language, $G0 = 0$ and $0G = 0$ for any matrix G, whenever the sizes of these zero matrices allow the products to be defined.