First Order Linear Differential Equations

The general first-order linear ordinary differential operator may be written formally as

$$L = a(x)D + b(x),$$

where D denotes d/dx. It acts on any differentiable function y of x to give

$$L(y) = a(x)\frac{dy}{dx} + b(x)y.$$

The homogeneous case The general *homogeneous* first-order linear ordinary differential equation may be written

$$L(y) \equiv a(x)\frac{dy}{dx} + b(x)y = 0. \tag{1}$$

It is separable, and so easily solved, and the solution always has the form $y = Au$, where A is an arbitrary constant.

Example Solve the differential equation

$$L(y) \equiv x^3\frac{dy}{dx} + 4x^2y = 0. \tag{2}$$

We separate the variables to get the equation of differentials

$$\frac{dy}{y} = -4\frac{dx}{x},$$

which integrates to give (with a little cleaning up)

$$\log y = -4\log x + C = \log(x^{-4}) + C.$$

We exponentiate this to give the solution in the desired form

$$y = e^{\log(x^{-4})}e^C = Ax^{-4}, \tag{3}$$

where we write the arbitrary constant as $A = e^C$.

By linearity, it is enough to find *one* nontrivial solution u of equation (2) and multiply it by the arbitrary constant A.

(continued on other side)

The nonhomogeneous case The general *nonhomogeneous* first-order linear ordinary differential equation may be written

$$L(y) \equiv a(x)\frac{dy}{dx} + b(x)y = Q(x). \tag{4}$$

We use the method of *variation of parameter(s).* The idea is to put $y(x) = A(x)u(x)$, where u is a solution of the corresponding homogeneous equation equation (1), except that the parameter A is now allowed to vary. By the product rule, $L(y)$ has some terms with $A(x)$ as a factor, and some terms with $A'(x)$ as a factor. The point is that *the terms with A must cancel,* because $L(Au) = 0$ if we choose A to be constant, by our choice of u. (This provides a useful check on our calculations.) Then equation (4) reduces to an equation for $A'(x)$, which is to be integrated to give $A(x)$ and hence the general solution y.

Example Solve the differential equation

$$L(y) \equiv x^3\frac{dy}{dx} + 4x^2y = x - 1. \tag{5}$$

In view of equation (3), we put $y(x) = A(x)x^{-4}$. Then

$$L(y) = x^3A'(x)x^{-4} - 4x^3A(x)x^{-5} + 4x^2A(x)x^{-4} = A'(x)x^{-1}.$$

Observe that the terms with $A(x)$ cancel. Thus equation (5) reduces to $A'(x) = x(x-1)$, which integrates to give $A(x) = x^3/3 - x^2/2 + C$. The general solution of equation (5) is therefore

$$y = \frac{A(x)}{x^4} = \frac{1}{3x} - \frac{1}{2x^2} + \frac{C}{x^4}, \tag{6}$$

where C is the arbitrary constant.