Integration on Manifolds and Lie Groups

This note fills in some details for §5 in Chapter I of Bröcker–tom Dieck.

First, we review some basic material on integration on manifolds.

Integration on a smooth manifold Let M be an *oriented* smooth n-manifold, and denote by $C_c^n(M)$ the vector space of continuous n-forms on M with compact support. Our object here is to define the integral $\int_M : C_c^n(M) \to \mathbb{R}$.

Take a smooth atlas on M consisting of *orientation-preserving* diffeomorphisms $h_\lambda : U_\lambda \cong U'_\lambda \subset \mathbb{R}^n$, where the open sets U_λ (for $\lambda \in \Lambda$) cover M. Take a partition of unity, consisting of continuous functions $\phi_\lambda : M \to \mathbb{R}$ with support $\operatorname{supp}(\phi_\lambda) \subset U_\lambda$, so that $\sum_\lambda \phi_\lambda = 1$. (We do not need ϕ_λ to be smooth for current purposes.)

Given $\omega \in C_c^n(M)$, we decompose it as the sum $\omega = \sum_\lambda \phi_\lambda \omega$; since every point of M has a neighborhood on which only finitely many of the ϕ_λ are nonzero and $\operatorname{supp}(\omega)$ is covered by finitely many such neighborhoods, this sum is effectively finite: all except finitely many terms are identically zero. We handle each term separately.

We pull back the n-form $\phi_\lambda \omega$ by h_λ^{-1} to the open set U'_λ in $\mathbb{R}^n$. The canonical orientation of $\mathbb{R}^n$ orients the standard basis $(e_1, e_2, \ldots, e_n)$ positively. We write the coordinates on $\mathbb{R}^n$ as $x = (x_1, x_2, \ldots, x_n)$ (and recall that the coordinate x_i is really a function, the projection $\mathbb{R}^n \to \mathbb{R}$ to the i-th factor, so that dx_i is a 1-form). The result is

$$(h_\lambda^{-1})^*(\phi_\lambda \omega) = a_\lambda \, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n,$$

where the function a_λ is given explicitly as

$$a_\lambda(x) = (a_\lambda \, dx_1 \wedge \ldots \wedge dx_n)_x(e_1, \ldots, e_n) = \phi_\lambda(p) \, \omega_p(T(h_\lambda^{-1})e_1, \ldots, T(h_\lambda^{-1})e_n) \tag{1}$$

and $p = h_\lambda^{-1}(x)$.

Definition 2 We define the integral in terms of multiple integrals on $\mathbb{R}^n$, as

$$\int_M \omega = \sum_\lambda \int \!\cdots\! \int_{U'_\lambda} a_\lambda(x) \, dx_1 \, dx_2 \ldots dx_n.$$

Remark If $\omega_p(v_1, v_2, \ldots, v_n) \geq 0$ for one (and hence any) positively oriented basis of the tangent space $T_p(M)$, equation (1) shows that $a_\lambda(h_\lambda(p)) \geq 0$. If the condition holds for all $p \in M$, we deduce that $\int_M \omega > 0$ unless ω is everywhere zero.

Remark If we reverse the orientation of M, the local chart h_λ no longer preserves orientation. We may replace it by $r \circ h_\lambda$, where r is the reflection of $\mathbb{R}^n$ that reverses the first coordinate x_1 only. Since $r^* dx_1 = dr^* x_1 = -dx_1$, we find

$$(h_\lambda^{-1} \circ r)^*(\phi_\lambda \omega) = r^*(a_\lambda \, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n) = -(r^* a_\lambda) \, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n.$$

However,

$$\int \!\cdots\! \int a_\lambda(r(x)) \, dx_1 \, dx_2 \ldots dx_n = \int \!\cdots\! \int a_\lambda(x) \, dx_1 \, dx_2 \ldots dx_n.$$

The net effect is to *change the sign* of $\int_M \omega$.

THEOREM 3 *Given an oriented smooth n-manifold M, Definition 2 yields a canonical $\mathbb{R}$-linear function*

$$\int_M : C_c^n(M) \longrightarrow \mathbb{R}.$$

Proof $\mathbb{R}$-linearity is obvious. We have to show that the integral is independent of the choices of atlas and partition of unity.

Take another atlas, consisting of orientation-preserving diffeomorphisms $k_\mu : V_\mu \cong V'_\mu \subset \mathbb{R}^n$, with associated partition of unity consisting of functions ψ_μ. We decompose $\omega = \sum_\lambda \sum_\mu \phi_\lambda \psi_\mu \omega$, so that $\phi_\lambda \psi_\mu \omega$ has compact support in $U_\lambda \cap V_\mu$. We pull back each term,

$$(h_\lambda^{-1})^*(\phi_\lambda \psi_\mu \omega) = a_{\lambda,\mu}\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n.$$

It is clear that $a_\lambda = \sum_\mu a_{\lambda,\mu}$. Similarly,

$$(k_\mu^{-1})^*(\phi_\lambda \psi_\mu \omega) = b_{\mu,\lambda}\, dy_1 \wedge dy_2 \wedge \ldots \wedge dy_n,$$

where we use coordinates $(y_1, y_2, \ldots, y_n)$ on this copy of $\mathbb{R}^n$ to avoid confusion. It will be sufficient to prove that

$$\int \cdots \int b_{\mu,\lambda}(y)\, dy_1\, dy_2 \ldots dy_n = \int \cdots \int a_{\lambda,\mu}(x)\, dx_1\, dx_2 \ldots dx_n.$$

We have the diffeomorphism $\theta = k_\mu \circ h_\lambda^{-1} : h_\lambda(U_\lambda \cap V_\mu) \cong k_\mu(U_\lambda \cap V_\mu)$ between open sets in $\mathbb{R}^n$, hence

$$\theta^*(b_{\mu,\lambda}\, dy_1 \wedge dy_2 \wedge \ldots \wedge dy_n) = a_{\lambda,\mu}\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n, \tag{4}$$

as both sides are pullbacks from $\phi_\lambda \psi_\mu \omega$ on $U_\lambda \cap V_\mu \subset M$. The left side is

$$(\theta^* b_{\mu,\lambda})(\theta^* dy_1) \wedge (\theta^* dy_2) \wedge \ldots \wedge (\theta^* dy_n).$$

Here, $(\theta^* b_{\mu,\lambda})(x) = b_{\mu,\lambda}(\theta(x))$. In coordinates, $\theta(x) = (\theta_1(x), \ldots, \theta_n(x)) \in \mathbb{R}^n$, where $\theta_j = y_j \circ \theta$. Then we can write

$$\theta^* dy_j = d\theta^* y_j = d(y_j \circ \theta) = d\theta_j = \sum_{i=1}^n \frac{\partial \theta_j}{\partial x_i}\, dx_i.$$

Properties of $\wedge$ show that $(\theta^* dy_1) \wedge (\theta^* dy_2) \wedge \ldots \wedge (\theta^* dy_n)$ expands to

$$J\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n,$$

where $J(x)$ denotes the Jacobian determinant of θ at x. Thus equation (4) reduces to $a_{\lambda,\mu}(x) = b_{\mu,\lambda}(\theta(x)) J(x)$.

Meanwhile, the change of variables theorem in $\mathbb{R}^n$ shows that

$$\begin{aligned}\int \cdots \int b_{\mu,\lambda}(y)\, dy_1\, dy_2 \ldots dy_n &= \int \cdots \int b_{\mu,\lambda}(\theta(x)) J(x)\, dx_1\, dx_2 \ldots dx_n \\ &= \int \cdots \int a_{\lambda,\mu}(x)\, dx_1\, dx_2 \ldots dx_n\end{aligned}$$

as required; since θ preserves the orientation of $\mathbb{R}^n$, $J > 0$ and we do not need to write $|J(x)|$ here. □

THEOREM 5 *Suppose $\theta\colon P \cong M$ is an orientation-preserving diffeomorphism of oriented smooth n-manifolds, and that ω is a continuous n-form on M. Then $\int_P \theta^*\omega = \int_M \omega$.*

Proof Take a smooth atlas $\{h_\lambda\colon U_\lambda \cong U'_\lambda \subset \mathbb{R}^n : \lambda \in \Lambda\}$ with partition of unity $\{\phi_\lambda : \lambda \in \Lambda\}$ as above. There is an obvious atlas on P, consisting of open sets $W_\lambda = \theta^{-1}(U_\lambda)$ with local charts

$$W_\lambda \xrightarrow{\theta} U_\lambda \xrightarrow{h_\lambda} U'_\lambda \subset \mathbb{R}^n,$$

and the functions $\phi_\lambda \circ \theta = \theta^*\phi_\lambda$ form a partition of unity on P. The pullback of $(\theta^*\phi_\lambda)(\theta^*\omega)$ on W_λ to U'_λ is

$$((h_\lambda \circ \theta)^{-1})^*((\theta^*\phi_\lambda)(\theta^*\omega)) = (h_\lambda^{-1})^*(\theta^{-1})^*\theta^*(\phi_\lambda\omega) = (h_\lambda^{-1})^*(\phi_\lambda\omega).$$

Our choices make $\int_P \theta^*\omega$ and $\int_M \omega$ identical, term by term. □

Integration on a Lie group From now on, we assume that G is a *compact* Lie group of dimension n. We wish to integrate a continuous real-valued function f over G. We write the integral as $\int_G f(g)\,dg$, where g denotes the variable of integration and dg should be considered a *measure* (instead of any kind of differential form) on G, known as *Haar measure*. (However, we avoid doing any measure theory.)

Denote by $C(G)$ the vector space of continuous real-valued functions on G. The integral is required to have the following four properties:

$$\begin{aligned} &\text{(i) It is an } \mathbb{R}\text{-}\textit{linear} \text{ function } C(G) \to \mathbb{R};\\ &\text{(ii) It is } \textit{monotone}\text{: if } f(g) \ge 0 \text{ for all } g, \text{ then } \textstyle\int_G f(g)\,dg \ge 0;\\ &\text{(iii) It is } \textit{left-invariant}\text{: for any } h \in G,\ \textstyle\int_G f(hg)\,dg = \int_G f(g)\,dg;\\ &\text{(iv) It is } \textit{normalized}\text{: } \textstyle\int_G 1\,dg = 1. \end{aligned} \tag{6}$$

THEOREM 7 *Given a compact Lie group G, there exists a unique integral $C(G) \to \mathbb{R}$ that satisfies the axioms* (6).

To prove existence, we treat G as an n-manifold and apply the preceding theory of integration of n-forms. (Uniqueness will be included in Corollary 13.)

First, we need to *orient* G, which is easy. We choose an orientation o_e of the tangent space $T_e(G)$. We use the left translations l_h to propagate this to an orientation on the whole of G: given any basis $\{v_1, v_2, \ldots, v_n\}$ of $T_h(G)$, for any $h \in G$, we define

$$o_h(v_1, v_2, \ldots, v_n) = o_e(T(l_{h^{-1}})v_1, T(l_{h^{-1}})v_2, \ldots, T(l_{h^{-1}})v_n).$$

This orientation is left-invariant: for any basis $\{v_1, v_2, \ldots, v_n\}$ of $T_g(G)$, we have

$$o_{hg}(T(l_h)v_1, T(l_h)v_2, \ldots, T(l_h)v_n) = o_g(v_1, v_2, \ldots, v_n),$$

and it is the only left-invariant orientation that extends o_e.

Next, we need an n-form on G. We choose a positively oriented alternating n-form ω_e on the vector space $T_e(G)$, and similarly propagate it around G by taking $\omega_h = l^*_{h^{-1}}\omega_e$ for all $h \in G$, so that ω is left-invariant, $l_h^*\omega = \omega$ for all $h \in G$.

DEFINITION 8 We define $\int_G f(g)\,dg = \int_G f\,m\omega$, where the number m is independent of f and is chosen to make (6)(iv) hold.

The axioms (6) are readily verified. Axiom (i) is clear. By the Remark following Definition 2, since ω is positively oriented everywhere, $\int \omega > 0$ and $m > 0$ can indeed be chosen to make axiom (iv) hold; further, axiom (ii) holds because $\int f\, m\omega \ge 0$ if $f \ge 0$ everywhere. For axiom (iii), we use $l_h^*\omega = \omega$ and Theorem 5 to express the left invariance as

$$\int f \circ l_h\, m\omega = \int l_h^* f\, m\omega = \int l_h^* f\, m l_h^*\omega = \int l_h^*(f\, m\omega) = \int f\, m\omega.$$

Remark If we reverse the orientation of G, we change ω to $-\omega$, but m stays the same. By the second Remark after Definition 2, the integral $\int f(g)\, dg$ *does not change.*

It is easy to construct a *right-invariant* integral, by introducing an inverse. It apparently requires a different measure, denoted by δg instead of dg.

Definition 9 We define the right-invariant integral

$$\int_G f(g)\, \delta g = \int_G f(g^{-1})\, dg. \tag{10}$$

This clearly is right-invariant,

$$\int f(gh)\, \delta g = \int f(h^{-1}g^{-1})\, dg = \int f(g^{-1})\, dg = \int f(g)\, \delta g.$$

The other three axioms are obvious.

We shall see later, in Corollary 13, that the two integrals in fact coincide, so that $\delta g = dg$.

To prove this, we need a Fubini-type result, which has nothing to do with Lie groups, or even manifolds. Let X and Y be compact metric spaces. Suppose we are given "integrals" $I_X \colon C(X) \to \mathbb{R}$ and $I_Y \colon C(Y) \to \mathbb{R}$ that satisfy axioms (i) and (ii) of (6). We define a partial integration over Y, $\widehat{I}_Y \colon C(X \times Y) \to C(X)$: given $F \colon X \times Y \to \mathbb{R}$, we define $\widehat{I}_Y F \in C(X)$ by $(\widehat{I}_Y F)(x) = I_Y F_x$, where $F_x \in C(Y)$ is given by $F_x(y) = F(x,y)$. We similarly define $\widehat{I}_X \colon C(X \times Y) \to C(Y)$.

Lemma 11 *For any* $F \in C(X \times Y)$, $I_X \widehat{I}_Y F = I_Y \widehat{I}_X F$.

Proof We observe that if we equip the spaces $C(X)$ etc. with the sup norm, the hypotheses imply that I_X and $\widehat{I}_X$ become bounded linear operators with norm $\mathrm{vol}(X)$, and I_Y and $\widehat{I}_Y$ have norm $\mathrm{vol}(Y)$, where we introduce the volumes $\mathrm{vol}(X) = I_X 1$ and $\mathrm{vol}(Y) = I_Y 1$.

By compactness, F is uniformly continuous. Given $\epsilon > 0$, choose $\delta > 0$ such that $d(x_1, x_2) < \delta$ and $d(y_1, y_2) < \delta$ imply that $|F(x_2, y_2) - F(x_1, y_1)| < \epsilon$. If $d(x_1, x_2) < \delta$, $\|F_{x_2} - F_{x_1}\| < \epsilon$ and $|(\widehat{I}_Y F)(x_2) - (\widehat{I}_Y F)(x_1)| < \epsilon\, \mathrm{vol}(X)$, which shows that $\widehat{I}_Y F$ does indeed lie in $C(X)$.

The result is obvious for a function of the form $F(x,y) = \phi(x)\psi(y)$; both sides reduce by $\mathbb{R}$-linearity to $(I_X\phi)(I_Y\psi)$. For a general F, we approximate by sums of such functions. We cover X by finitely many open sets U_λ of diameter less than δ, take a partition of unity consisting of functions ϕ_λ with support in U_λ, and choose points $x_\lambda \in U_\lambda$; similarly, we cover Y by open sets V_μ with functions ψ_μ and points y_μ. We may then write

$$F(x,y) = \sum_\lambda \sum_\mu \phi_\lambda(x)\psi_\mu(y)F(x,y).$$

We approximate F by the function

$$E(x,y) = \sum_\lambda \sum_\mu \phi_\lambda(x)\psi_\mu(y)F(x_\lambda,y_\mu);$$

then

$$|F(x,y) - E(x,y)| \le \sum_\lambda \sum_\mu \phi_\lambda(x)\psi_\mu(y)\,|F(x,y) - F(x_\lambda,y_\mu)|.$$

For $\phi_\lambda(x)$ and $\psi_\mu(y)$ to be both nonzero, we must have $x \in U_\lambda$ and $y \in V_\mu$, hence $|F(x_\lambda,y_\mu) - F(x,y)| < \epsilon$. This yields, for all x and y,

$$|F(x,y) - E(x,y)| \le \sum_\lambda \sum_\mu \phi_\lambda(x)\psi_\mu(y)\epsilon = \epsilon.$$

Thus $\|F - E\| \le \epsilon$, which implies that $|I_X\widehat{I}_YF - I_X\widehat{I}_YE| \le \epsilon\,\mathrm{vol}(X)\mathrm{vol}(Y)$ and $|I_Y\widehat{I}_XF - I_Y\widehat{I}_XE| \le \epsilon\,\mathrm{vol}(X)\mathrm{vol}(Y)$. But E is an $\mathbb{R}$-linear combination of the functions $\phi_\lambda(x)\psi_\mu(y)$, for which we have $I_X\widehat{I}_YE = I_Y\widehat{I}_XE$. Then $|I_X\widehat{I}_YF - I_Y\widehat{I}_XF| \le 2\epsilon\,\mathrm{vol}(X)\mathrm{vol}(Y)$. Since ϵ is arbitrary, we must have $I_X\widehat{I}_YF = I_Y\widehat{I}_XF$. □

We apply this to the case $X = Y = G$.

THEOREM 12 *Let I_Y be any integral on G that satisfies axioms* (6), *in particular, left invariance, and I_X be any right-invariant integral on G that satisfies the axioms* (*with* (iii) *modified*)*. Then $I_Y = I_X$.*

COROLLARY 13 *There are exactly one left-invariant integral and one right-invariant integral on G, and they are equal.* □

Remark Dealing directly with n-forms and orientations is more complicated and is best avoided. A left-invariant n-form ω on G is *not* right-invariant in general, unless G is connected; instead, the n-form $\nu^*\omega$, where $\nu\colon G \to G$ denotes inversion, is right-invariant, but satisfies $\nu^*\omega = \pm\omega$, where the sign can vary from component to component of G.

Orientations behave the same way.

Proof of Theorem Given $f \in C(G)$, we consider $F(g,h) = f(gh)$. We note that $F_g(h) = f(gh) = (f \circ l_g)(h)$, or $F_g = f \circ l_g$. Integrating over h, we have

$$(\widehat{I}_YF)(g) = I_YF_g = I_Y(f \circ l_g) = I_Yf,$$

which is independent of g. Then $I_X\widehat{I}_YF = I_X(I_Yf.1) = (I_Yf)(I_X1) = I_Yf$. Similarly, $I_Y\widehat{I}_XF = I_Xf$. □

THEOREM 14 *Let G be a compact Lie group and $f \in C(G)$. Then for any $h \in G$ and any automorphism ϕ of G, we have*

$$\int f(g)\,dg = \int f(hg)\,dg = \int f(gh)\,dg = \int f(g^{-1})\,dg = \int f(\phi(g))\,dg.$$

Proof The first two integrals are equal by left invariance. The fourth is our definition of the right-invariant integral, which we have just seen is equal to the first. So the left-invariant integral is also right-invariant, which gives the third integral.

For the fifth, a different strategy is needed. We define the integral $If = \int f(\phi(g))\,dg$. It clearly satisfies axioms (i), (ii) and (iv). For (iii), we have

$$\begin{aligned} I(f \circ l_h) &= \int f(h\phi(g))\,dg = \int f(\phi(\phi^{-1}(h)g))\,dg = \int (f\circ\phi)(\phi^{-1}(h)g)\,dg \\ &= \int (f\circ\phi)(g)\,dg \quad \text{by left invariance, (6)(iii)} \\ &= \int f(\phi(g))\,dg = If. \end{aligned}$$

By uniqueness, If must coincide with the first integral, $\int f(g)\,dg$. □