G-sets, G-spaces and Covering Spaces

These notes amplify pages 68–72 of Hatcher's "Algebraic Topology".

Here, G will always be a *discrete* group (though the definitions make sense and are useful for topological groups, also with spaces replaced by many other kinds of mathematical object).

G-spaces

DEFINITION 1 A (left) *G-space* Y is a space Y equipped with an *action* map $G\times Y \to Y$, usually written $(g,y)\mapsto gy$, that satisfies the axioms

$$\begin{aligned} &\text{(i)}\ \ 1y = y \text{ for any } y\in Y;\\ &\text{(ii)}\ \ g(g'y) = (gg')y \text{ for any } y\in Y,\ g,g'\in G. \end{aligned} \tag{2}$$

The *orbit* of $y\in Y$ is the subspace $Gy=\{gy\in Y : g\in G\}$. The *stabilizer* of $y\in Y$ is the subgroup $H_y=\{g\in G : gy=y\}$ of G.

If Z is another G-space, a map $f\colon Y\to Z$ is a *G-map* if $f(gy)=g(f(y))$ for all $y\in Y$ and $g\in G$.

In view of (ii), we may omit many parentheses. For each $g\in G$, the map $y\mapsto gy$ is a homeomorphism of Y, usually written simply as $g\colon Y\to Y$, with inverse homeomorphism $g^{-1}\colon Y\to Y$.

The axioms imply that H_y is indeed a subgroup of G (proof omitted). We have the useful identity

$$H_{gy} = gH_yg^{-1}. \tag{3}$$

To see this, note that $gH_yg^{-1}\subset H_{gy}$ because for any $h\in H_y$, $(ghg^{-1})gy = ghy = gy$. Similarly, we have $g^{-1}H_{gy}g\subset H_y$, which implies $H_{gy}\subset gH_yg^{-1}$.

If $f\colon Y\to Z$ is a G-map, it is clear that $H_y\subset H_{f(y)}$ for any $y\in Y$.

LEMMA 4 *The orbits in a G-space Y form a partition of Y: $y\in Gy$ for any $y\in Y$, and for any $y,z\in Y$, either $Gy=Gz$ or Gy and Gz are disjoint.*

Proof Clearly, $y=1y\in Gy$. Suppose $w\in Gy\cap Gz$, i.e. $w=gy=hz$ for some $g,h\in G$. Then $g^{-1}w=g^{-1}gy=1y=y$. Given any $k\in G$, $ky=kg^{-1}w=kg^{-1}hz\in Gz$ shows that $Gy\subset Gz$; similarly, $Gz\subset Gy$. □

DEFINITION 5 Given a G-space Y, the *orbit space* Y/G of Y is the set of all orbits in Y, topologized as a quotient space of Y by means of the obvious quotient map $q\colon Y\to Y/G$ given by $q(y)=Gy$.

The G-space Y is *free* if for every $y\in Y$, $gy\neq y$ for all $g\neq 1$ in G. This implies that as g varies, the points gy are all distinct. We consider a stronger condition:

$$\text{Every point } y\in Y \text{ has a neighborhood } U \text{ such that } U\cap gU=\emptyset \text{ for all } g\neq 1. \tag{6}$$

Then as g varies, the sets gU are all disjoint and open. It follows that $q\colon Y\to Y/G$ is a covering map: since $q^{-1}(q(U))=\coprod_g gU$, the set $GU=q(U)$ is open in Y/G and is evenly covered by the sets $gU\subset Y$.

G-sets A *G-set* is simply a discrete G-space. It is *transitive* if there is only one orbit.

Example Any disjoint union of G-sets is a G-set.

Example Given a subgroup $H \subset G$, denote by G/H the set of all cosets $gH = \{gh : h \in H\}$ of H in G. It is a transitive G-set with action $g'(gH) = (g'g)H$. (This does not conflict with the notation of Definition 5: G/H is the orbit space of the action of H on G given by *right* multiplication, $(h, g) \mapsto gh^{-1}$.)

The stabilizer of the coset $H = 1H \in G/H$ is just $H_H = H$. Observe that the number of elements of G/H is the index of H in G, if finite.

G-sets are easily classified. We note that each orbit is itself a G-set.

THEOREM 7 *Let G be a discrete group.*

(a) *Any G-set Y is the disjoint union of its orbits;*

(b) *For any $y \in Y$, the orbit Gy is isomorphic to the G-set G/H_y;*

(c) *The G-sets G/H and G/K are isomorphic if and only if the subgroups H and K of G are conjugate.*

Proof Lemma 4 takes care of (a).

For (b), we map $\phi: G/H_y \to Gy$ by $\phi(gH_y) = gy$; since $ghy = gy$ for any $h \in H_y$, this is well defined. It is visibly surjective. Suppose that $\phi(gH_y) = \phi(kH_y)$, i. e. $gy = ky$. Then $k^{-1}gy = y$, $k^{-1}g \in H_y$, and $kH_y = k(k^{-1}g)H_y = gH_y$.

Equation (3) gives the necessity in (c). Conversely, we define the isomorphism of G-sets $G/H \cong G/gHg^{-1}$ by $kH \mapsto kHg^{-1} = (kg^{-1})(gHg^{-1})$ for any $k \in G$. □

We need to determine the automorphism group of G/H. With that in hand, we can write down the automorphism group of any G-set.

THEOREM 8 *Given a subgroup H of G, the automorphism group $\mathrm{Aut}(G/H)$ of the G-set G/H is isomorphic to the group $N(H)/H$, where $N(H) = \{g \in G : gHg^{-1} = H\}$ is the* normalizer *of H in G, the largest subgroup of G in which H is a normal subgroup.*

The action of $\mathrm{Aut}(G/H)$ on G/H is transitive if and only if H is normal in G.

Proof Take any G-map $\theta: G/H \to G/H$. Then $\theta(H) = k^{-1}H$ for some $k \in G$. (The reason to use k^{-1} here will appear later.) Because θ is a G-map, we have

$$\theta(gH) = g\theta(H) = gk^{-1}H \qquad \text{for any } g \in G, \tag{9}$$

so that the element $k \in G$ completely determines θ. Now take $g = h \in H$; since $hH = H$, we must have $hk^{-1}H = k^{-1}H$, which reduces to $khk^{-1} \in H$, hence $kHk^{-1} \subset H$. This is enough to ensure that equation (9) is well defined for all g, as

$$\theta(ghH) = ghk^{-1}H = gk^{-1}(khk^{-1})H = gk^{-1}H = \theta(gH).$$

We note that θ is automatically surjective, as $\theta(gkH) = gkk^{-1}H = gH$ for any $g \in G$. However, it is *not* automatically 1-1. Suppose $\theta(gH) = \theta(H)$. Then $gk^{-1}H = k^{-1}H$, which reduces to $kgk^{-1} \in H$, then to $g \in k^{-1}Hk$. For θ to be 1-1, this must imply $g \in H$, i.e. $k^{-1}Hk \subset H$, or $H \subset kHk^{-1}$. This, together with $kHk^{-1}H \subset H$, shows that $k \in N(H)$.

The action of $\mathrm{Aut}(G/H)$ on G/H is transitive if and only if $\theta(H) = k^{-1}H$ can be any element of G/H. Since $k \in N(H)$ is arbitrary, this is equivalent to $N(H) = G$.

Thus given $k \in N(H)$, we have the automorphism θ_k of G/H defined by (9). Now $k \mapsto \theta_k$ is a homomorphism of groups from $N(H)$ to $\mathrm{Aut}(G/H)$, because

$$\theta_m(\theta_k(gH)) = \theta_m(gk^{-1}H) = gk^{-1}m^{-1}H = \theta_{mk}(gH).$$

We saw that this homomorphism is surjective, and its kernel is clearly H. □

Applications to covering spaces

PROPOSITION 10 *Let $p\colon \tilde{X} \to X$ be a covering map, and put $\pi = \pi_1(X, x_0)$. Then the fibre $F = p^{-1}(x_0)$ is a π-set, and the stabilizer of any point $\tilde{x} \in F$ is $H_{\tilde{x}} = p_*(\pi_1(\tilde{X}, \tilde{x}))$.*

Proof Given a point $\tilde{x} \in F$ and a loop γ in X at x_0, lift γ to a path $\tilde{\gamma}$ *ending* at $\tilde{x}$. We define $[\gamma]\tilde{x} = \tilde{\gamma}(0) \in F$; it is well defined, because if $\gamma \simeq \delta$, $\tilde{\gamma} \simeq \tilde{\delta}$. If $[\gamma] \in H_{\tilde{x}}$, $\tilde{\gamma}$ is a loop, and $[\gamma] = p_*[\tilde{\gamma}]$; and conversely.

We must verify the axioms (2). If γ is the constant path c_{x_0} at x_0, $\tilde{\gamma} = c_{\tilde{x}}$, and we have $1\tilde{x} = [c_{x_0}]\tilde{x} = c_{\tilde{x}}(0) = \tilde{x}$. Let β be another loop at x_0. We lift it to a path $\tilde{\beta}$ ending at $\tilde{\gamma}(0)$, so that $[\beta]([\gamma]\tilde{x}) = [\beta]\tilde{\gamma}(0) = \tilde{\beta}(0)$. Then $\beta \cdot \gamma$ lifts to $\tilde{\beta} \cdot \tilde{\gamma}$, and $([\beta][\gamma])\tilde{x} = [\beta \cdot \gamma]\tilde{x} = (\tilde{\beta} \cdot \tilde{\gamma})(0) = \tilde{\beta}(0)$. □

Remark Some writers avoid the contortions in this proof by making the action of π on F a *right* action.

COROLLARY 11 *Assume that X is path-connected. Then each path component of $\tilde{X}$ intersects the π-set F in exactly one orbit. In particular, $\tilde{X}$ is path-connected if and only if F is a transitive π-set.*

Proof First, each orbit in F lies in one path-component of $\tilde{X}$. If $\tilde{x}_1$ and $\tilde{x}_2$ belong to the same orbit, say $\tilde{x}_2 = [\gamma]\tilde{x}_1$, the loop γ lifts to a path in $\tilde{X}$ from $\tilde{x}_2$ to $\tilde{x}_1$.

Conversely, if $\tilde{x}_1$ and $\tilde{x}_2$ are two points of F that lie in the same path-component of $\tilde{X}$, there is a path β in $\tilde{X}$ from $\tilde{x}_2$ to $\tilde{x}_1$, hence $[p \circ \beta]\tilde{x}_1 = \tilde{x}_2$, and the two points are in the same orbit.

Finally, each path component of $\tilde{X}$ contains a point of F. Given $\tilde{x} \in \tilde{X}$, choose a path β in X from $p(\tilde{x})$ to x_0, and lift it to a path $\tilde{\beta}$ in $\tilde{X}$ from $\tilde{x}$ to some point $\tilde{\beta}(1) \in F$. □

COROLLARY 12 *If X and $\tilde{X}$ are path-connected, the number of sheets in the covering is the index in π of the stabilizer $H_{\tilde{x}}$ of any point $\tilde{x} \in F$.* □

Now suppose that $p_1\colon \tilde{X}_1 \to X$ and $p_2\colon \tilde{X}_2 \to X$ are two covering maps, with fibres $F_1 = p_1^{-1}(x_0)$ and $F_2 = p_2^{-1}(x_0)$. It is obvious that a map $f\colon \tilde{X}_1 \to \tilde{X}_2$ *over* X (one that satisfies $p_2 \circ f = p_1$) induces a π-map of π-sets $f|F_1\colon F_1 \to F_2$, where $\pi = \pi_1(X, x_0)$. We have the converse, for well-behaved X.

THEOREM 13 *Assume that X is path-connected and locally path-connected. Then any π-map $F_1 \to F_2$ extends uniquely to a map $f\colon \tilde{X}_1 \to \tilde{X}_2$ of covering spaces over X.*

Proof As $\tilde{X}_1$ is a covering space of X, it too is locally path-connected, and its path components are therefore open. It suffices to construct f on each path component of $\tilde{X}_1$, which reduces the problem to the known case where $\tilde{X}_1$ is path-connected. □

A homeomorphism of $\tilde{X}$ over X is known as a *deck transformation*. These form the automorphism group $\mathrm{Aut}(\tilde{X})$ of $\tilde{X}$. We combine Theorems 13 and 8.

COROLLARY 14 *With X as in Theorem 13, suppose that $\tilde{X}$ is a path-connected covering space and $\tilde{x}_0 \in F = p^{-1}(x_0)$. Then the group $\mathrm{Aut}(\tilde{X})$ of deck transformations of $\tilde{X}$ is isomorphic to $N(H)/H$, where $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ and $N(H)$ denotes the normalizer of H in π.*

The action of $\mathrm{Aut}(\tilde{X})$ on F is transitive if and only if H is normal in π. If so, we have a regular *or* normal *covering space, π/H is a group, and we can identify X with the orbit space $\tilde{X}/(\pi/H)$.* □

Remark Even if H is normal in π, the π-action (via π/H) on $\tilde{X}$ given by Corollary 14 is unrelated to the π-action on the fibre F given by Proposition 10 (unless π/H happens to be abelian); in particular, the action on $\tilde{X}$ does *not* in general extend the action on F. Write $q\colon \pi \to \pi/H$ for the quotient homomorphism; then in terms of the proof of Theorem 8, the π-set action on F is given by $g'q(g) = q(g')q(g)$, using left multiplication, while by equation (9), the deck transformation action induces on $F \cong \pi/H$ the action $\theta_k q(g) = q(g)q(k^{-1})$, using right multiplication.

Universal covering spaces A simply-connected covering space of X is called a *universal covering space* of X. Under the conditions of Theorem 13, it is unique up to homeomorphism. A further condition is needed to ensure existence.

COROLLARY 15 *With X as in Theorem 13, suppose that $\tilde{X}$ is a universal covering space. Then $\tilde{X}$ has the group π of deck transformations, and given any subgroup H of π, the orbit space $\tilde{X}/H$ is a covering space of X with $\pi_1(\tilde{X}/H) \cong H$.*

Proof As the subgroup $\{1\}$ of π is normal, we have the group π of deck transformations of $\tilde{X}$. Then $p\colon \tilde{X} \to X$ factors to give a covering map $p'\colon \tilde{X}/H \to X$, and $p'^{-1}(x_0)$ is the π-set π/H, with stabilizer group $H \cong \pi_1(\tilde{X}/H)$. □