Graded Algebra

We reinterpret the concepts of cycles, cocycles, products etc., in terms of chain maps of nonzero degree and chain homotopies between them.

Graded groups A *graded group* A is just a family of abelian groups A_i, one for each integer i. An element $a \in A_i$ is said to have *degree* i, and we write $|a| = i$. By abuse of language, we sometimes write $a \in A$ to mean $a \in A_i$ for some i when the specific i is not of interest. (Here, we deal only with $\mathbb{Z}$-graded groups, although it is possible, and occasionally useful, for gradings to take values in other (abelian) groups G.)

Remark The graded group A should not be confused with the direct sum $\bigoplus_i A_i$ or direct product $\prod_i A_i$, which may be constructed if desired; however, there is generally little or no interest in adding elements of different degrees.

There is an obvious definition of a homomorphism $f\colon A \to B$ of graded groups, but we need to be more general. A *homomorphism of graded groups $f\colon A \to B$ of degree n* consists of a family of homomorphisms $f_i\colon A_i \to B_{i+n}$, one for each $i \in \mathbb{Z}$. These compose in the obvious way: if also $g\colon B \to C$ has degree m, the composite $h = g \circ f\colon A \to C$ has components $h_i = g_{i+n} \circ f_i\colon A_i \to C_{i+n+m}$ and has degree $n + m$. We thus have the *graded* category of graded groups; we generally suppress many degree suffixes by working in this category.

Tensor products Given two graded groups A and B, their *tensor product* $A \otimes B$ is the graded group with components $(A\otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j$. Thus $(A\otimes B)_n$ is generated by elements $a \otimes b$, where $a \in A$, $b \in B$, and $|a| + |b| = n$. This construction is clearly commutative and associative, $A\otimes B \cong B\otimes A$ and $(A\otimes B)\otimes C \cong A\otimes(B\otimes C)$, under the obvious correspondences $a \otimes b \leftrightarrow b \otimes a$ and $(a\otimes b) \otimes c \leftrightarrow a \otimes (b\otimes c)$.

For reasons which will become apparent, if we are given two homomorphisms of graded groups, $f\colon A \to A'$ of degree m and $g\colon B \to B'$ of degree n, we define their tensor product homomorphism $f \otimes g\colon A \otimes B \to A' \otimes B'$ by the formula

$$(f \otimes g)(a \otimes b) = (-1)^{|g||a|} fa \otimes gb. \tag{1}$$

Thus $f\otimes g$ has degree $m+n$, as expected. If we also have homomorphisms $f'\colon A' \to A''$ and $g'\colon B' \to B''$, we can readily check the composition rule

$$(f' \otimes g') \circ (f \otimes g) = (-1)^{|g'||f|}(f' \circ f) \otimes (g' \circ g). \tag{2}$$

A *graded algebra* A is a graded group A equipped with (at least) a multiplication homomorphism $\mu\colon A \otimes A \to A$ of degree 0. One example is the cohomology ring $H^*(X)$ of a space X, which satisfies the identity

$$yx = (-1)^{|x||y|}xy. \tag{3}$$

Standard signs In view of equations (1), (2) and (3) and other examples which follow, it is customary and convenient in algebraic topology to adopt the following sign convention:

Whenever the order of two symbols of odd degree is reversed, a minus sign − is introduced.

Note that this is a purely lexical convention; it depends only on the order in which symbols appear, not on their meanings or other properties. It works best when the same graded symbols appear exactly once each on both sides of an equation. It is automatically conserved under various elementary manipulations of equations. But if a symbol with odd degree appears twice on the same side of an equation, the convention becomes inapplicable.

Because examples such as the ring $H^*(X)$ occur frequently, a graded algebra A that satisfies (3) is often called *commutative in the graded sense*, or *graded-commutative*, or (potentially confusingly) just *commutative*.

Chain complexes A *chain complex* A is a graded group A equipped with a homomorphism $\partial\colon A \to A$ (or occasionally ∂_A) of degree -1 that satisfies $\partial \circ \partial = 0$; ∂ is called the *boundary operator* or *differential*.

Given chain complexes A and B, we wish to make their tensor product graded group $A \otimes B$ a chain complex too; however, the obvious definition $\partial(a \otimes b) = \partial a \otimes b + a \otimes \partial b$ fails, as it implies $\partial\partial(a \otimes b) = 2(\partial a \otimes \partial b) \neq 0$. In order to force the two unwanted terms to cancel, it is customary to introduce a sign, suggested by the above sign convention, as follows:

$$\partial(a \otimes b) = \partial a \otimes b + (-1)^{|a|} a \otimes \partial b. \tag{4}$$

This cannot be done in a completely symmetrical way. In view of (1), this can be written simply as

$$\partial_{A \otimes B} = \partial_A \otimes \mathrm{id}_B + \mathrm{id}_A \otimes \partial_B. \tag{5}$$

Nevertheless, the tensor product construction *is* commutative if we are willing to enlarge our concept of commutativity.

PROPOSITION 6 *Given two chain complexes A and B, there is a natural isomorphism $T\colon A \otimes B \cong B \otimes A$ of chain complexes, defined by $T(a \otimes b) = (-1)^{ij}\, b \otimes a$, where $a \in A_i$ and $b \in B_j$.*

Proof We compare

$$\partial T(a \otimes b) = (-1)^{ij} \partial(b \otimes a) = (-1)^{ij} \partial b \otimes a + (-1)^{ij+j} b \otimes \partial a$$

and

$$\begin{aligned} T\partial(a \otimes b) &= T(\partial a \otimes b) + (-1)^i T(a \otimes \partial b) \\ &= (-1)^{(i-1)j} b \otimes \partial a + (-1)^{i+i(j-1)} \partial b \otimes a. \end{aligned}$$

We see that the signs agree. □

Associativity does not present a problem.

PROPOSITION 7 *Given chain complexes A, B and C, there is an obvious canonical isomorphism $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ of chain complexes, given by $a \otimes (b \otimes c) \leftrightarrow (a \otimes b) \otimes c$.*

Proof If $a \in A_i$, $b \in B_j$ and $c \in C_k$, we have

$$\begin{aligned} \partial(a \otimes (b \otimes c)) &= \partial a \otimes (b \otimes c) + (-1)^i a \otimes \partial(b \otimes c) \\ &= \partial a \otimes (b \otimes c) + (-1)^i a \otimes (\partial b \otimes c) + (-1)^{i+j} a \otimes (b \otimes \partial c) \end{aligned}$$

and

$$\begin{aligned}\partial((a\otimes b)\otimes c) &= \partial(a\otimes b)\otimes c + (-1)^{i+j}(a\otimes b)\otimes \partial c\\ &= (\partial a\otimes b)\otimes c + (-1)^i(a\otimes \partial b)\otimes c + (-1)^{i+j}(a\otimes b)\otimes \partial c,\end{aligned}$$

which clearly correspond as required. □

Chain maps and chain homotopy Given chain complexes A and B, a *chain map* $f: A \to B$ *of degree* n is a homomorphism of degree n that satisfies the identity $\partial \circ f = (-1)^{|f|} f \circ \partial$, as suggested by the sign convention. It is easy to check that if $g: B \to C$ is another chain map, of degree m, the composite $g \circ f: A \to C$ is a chain map of degree $m+n$. Also, if $f: A \to A'$ and $g: B \to B'$ are chain maps of degrees m and n, the homomorphism $f \otimes g: A \otimes B \to A' \otimes B'$ defined by (1) is a chain map of degree $m+n$.

Two chain maps $f, g: A \to B$ of degree n are called *chain homotopic* if there is a graded homomorphism $s: A \to B$ of degree $n+1$ that satisfies the identity

$$\partial \circ s + (-1)^n s \circ \partial = g - f. \tag{8}$$

Then s is called a *chain homotopy from* f *to* g, and we may write $s: f \simeq g: A \to B$.

The sign in (8) is clearly needed to make the following result true.

LEMMA 9 *Suppose* $f \simeq g: A \to B$ *are chain homotopic chain maps of degree* n. *Then:*

(a) *Chain homotopy is an equivalence relation.*

(b) *If* $k: C \to A$ *is a chain map, then* $f \circ k \simeq g \circ k: C \to B$.

(c) *If* $h: B \to C$ *is a chain map, then* $h \circ f \simeq h \circ g: A \to C$.

Proof The usual proof of (a) when $n = 0$ carries over to the general case.

Let s be a chain homotopy as in (8). Then in (b), composing (8) with k yields

$$\begin{aligned}g\circ k - f\circ k &= \partial\circ s\circ k + (-1)^n s\circ\partial\circ k\\ &= \partial\circ(s\circ k) + (-1)^{n+|k|}(s\circ k)\circ\partial,\end{aligned}$$

which shows that $s \circ k$ is a chain homotopy from $f \circ k$ to $g \circ k$.

Similarly for (c), composing h with (8) yields

$$\begin{aligned}h\circ g - h\circ f &= h\circ\partial\circ s + (-1)^n h\circ s\circ\partial\\ &= (-1)^{|h|}\partial\circ(h\circ s) + (-1)^n(h\circ s)\circ\partial,\end{aligned}$$

which shows that $(-1)^{|h|} h \circ s$ is the desired chain homotopy. □

The dictionary The main ingredients are:

(i) The singular chain complex $C(X)$ of a space X;

(ii) The chain map $f_\#: C(X) \to C(Y)$ induced by a map $f: X \to Y$;

(iii) The chain complex $\underline{G}$ that has G in degree zero and 0 elsewhere, where G is any (abelian) group;

(iv) The augmentation chain map $\epsilon\colon C(X) \to \underline{\mathbb{Z}}$.

We now reinterpret various standard items in homology and cohomology theory in terms of these ideas.

An n**-chain** $c \in C_n(X)$ corresponds to the homomorphism $c\colon \underline{\mathbb{Z}} \to C(X)$ of degree n determined by $1 \mapsto c$. If $f\colon X \to Y$ is a map, the n-chain $f_\# c \in C_n(Y)$ corresponds to the homomorphism

$$f_\# c\colon \underline{\mathbb{Z}} \xrightarrow{c} C(X) \xrightarrow{f_\#} C(Y),$$

which is a chain map if c is a chain map. By Lemma 9(c), if we change c by a boundary (or chain homotopy), $f_\# c$ changes by a boundary (or chain homotopy).

The **boundary** ∂c of the n-chain c corresponds to the homomorphism

$$\partial c\colon \underline{\mathbb{Z}} \xrightarrow{c} C(X) \xrightarrow{\partial} C(X),$$

which has degree $n-1$. It clearly sends $1 \in \mathbb{Z}$ to the element $\partial c \in C_{n-1}(X)$.

The n-chain c is an n**-cycle** if the corresponding homomorphism $c\colon \underline{\mathbb{Z}} \to C(X)$ is a *chain map*; as $c \circ \partial = c \circ 0 = 0$ here, this means that $\partial \circ c = 0\colon \underline{\mathbb{Z}} \to C(X)$.

The n-chain c is a n**-boundary** if it corresponds to a homomorphism of the form $\partial b = \partial \circ b$, for some homomorphism $b\colon \underline{\mathbb{Z}} \to C(X)$ of degree $n+1$.

Two n-cycles a and b on X are **homologous** if the corresponding chain maps $a, b\colon \underline{\mathbb{Z}} \to C(X)$ are *chain homotopic*. So a homology class on X corresponds to a chain homotopy class of chain maps that correspond to cycles on X.

Given a pair of spaces (X, A), a **relative** n**-chain** c corresponds to a homomorphism $c\colon \underline{\mathbb{Z}} \to C(X, A) = C(X)/C(A)$ of degree n. Hence relative n-cycles, relative n-boundaries, etc.

An n**-chain** $c \in C_n(X; G)$ **with coefficients in** G corresponds to a homomorphism $c\colon \underline{\mathbb{Z}} \to C(X) \otimes \underline{G}$ of degree n. Hence n-cycles, n-boundaries, etc., *with coefficients in* G.

An n**-cochain** ϕ on X **with coefficients in** G corresponds to the homomorphism $\phi\colon C(X) \to \underline{G}$ given by $c \mapsto \langle \phi, c\rangle \in G$ (where we understand $\langle \phi, c\rangle = 0$ if $|c| \neq n$), which has degree $-n$ (not n). If $f\colon Y \to X$ is a map, the cochain $f^\# \phi \in C^n(Y; G)$ corresponds to the homomorphism

$$f^\# \phi\colon C(Y) \xrightarrow{f_\#} C(X) \xrightarrow{\phi} G,$$

which is a chain map if ϕ is a chain map. If we change ϕ by a chain homotopy, Lemma 9(b) shows that $f^\# \phi$ also changes by a chain homotopy.

The **coboundary** $\delta\phi$ of the n-cochain ϕ corresponds to the homomorphism $\phi \circ \partial\colon C(X) \to G$, which has degree $-n-1$. If c is an $(n+1)$-chain,

$$\langle \delta\phi, c\rangle = \langle \phi, \partial c\rangle. \tag{10}$$

An n**-cocycle** ϕ on X (with coefficients in G) corresponds to a *chain map* $C(X) \to \underline{G}$ of degree $-n$, etc; as $\partial \circ \phi = 0$ here, this means that $\phi \circ \partial = 0$.

The **unit element** $1 \in C^0(X; \mathbb{Z})$ corresponds to the *augmentation* chain map $\epsilon\colon C(X) \to \underline{\mathbb{Z}}$. Its cohomology class $1 \in H^0(X; \mathbb{Z})$, the unit element of the ring $H^*(X; \mathbb{Z})$, corresponds to the chain homotopy class of ϵ.

Two n-cocycles ϕ and ψ on X are **cohomologous** if the corresponding chain maps $\phi, \psi: C(X) \to \underline{G}$ are *chain homotopic.* So a cohomology class on X corresponds to a chain homotopy class of chain maps that correspond to cocycles on X.

The n-cochain ϕ on X is a **coboundary** if the corresponding homomorphism $C(X) \to \underline{G}$ has the form $\delta\psi = \psi \circ \partial$ for some homomorphism $\psi: C(X) \to \underline{G}$ (which must have degree $-n+1$, and so corresponds to a $(n-1)$-cochain).

A **relative n-cochain** ϕ on the pair (X, A) corresponds to a homomorphism $\phi: C(X) \to \underline{G}$ of degree $-n$ that maps $C(A)$ to zero, and so factors through $C(X)/C(A)$. Hence relative n-cocycles, relative n-coboundaries, etc.

Products We expand our dictionary by reinterpreting the various products. We assume that R is a commutative (ungraded) ring with multiplication $\mu: R \otimes R \to R$, which induces the obvious chain map $\underline{\mu}: \underline{R} \otimes \underline{R} \to \underline{R}$. (For simplicity, we often take $R = \mathbb{Z}$ here, leaving the general case as an exercise in inserting $\underline{R}$ as needed.)

We need the natural chain homotopy equivalences

$$\alpha(X, Y): C(X \times Y) \longrightarrow C(X) \otimes C(Y),$$
$$\beta(X, Y): C(X) \otimes C(Y) \longrightarrow C(X \times Y),$$

that are provided by acyclic model theory or explicit formulae. They are well defined up to chain homotopy and are chain homotopy inverses.

We also need the diagonal map $\Delta: X \to X \times X$.

The **scalar product** $\langle \phi, c \rangle \in \mathbb{Z}$ of a cochain $\phi \in C^k(X)$ and a chain $c \in C_k(X)$ corresponds to the homomorphism

$$\langle \phi, c \rangle: \underline{\mathbb{Z}} \xrightarrow{\ c\ } C(X) \xrightarrow{\ \phi\ } \underline{\mathbb{Z}} \tag{11}$$

of degree zero, which is multiplication by the integer $\langle \phi, c \rangle$. This homomorphism is trivially a chain map. (If $c \in C_l(X)$ where $l \neq k$, (11) defines $\langle \phi, c \rangle$ as trivially zero.) If ϕ and c are chain maps, Lemma 9 shows that if we change either (or both) by a chain homotopy, $\langle \phi, c \rangle$ also changes by a chain homotopy. Here, chain homotopic chain maps $\underline{\mathbb{Z}} \to \underline{\mathbb{Z}}$ are automatically equal, so the scalar product $\langle \phi, c \rangle$ does not change.

The **cup product** $\phi \cup \psi \in C^{k+l}(X; R)$ of two cochains $\phi \in C^k(X; R)$ and $\psi \in C^l(X; R)$ corresponds to the homomorphism

$$\begin{aligned} \phi \cup \psi: C(X) \xrightarrow{\ \Delta_\#\ } C(X \times X) &\xrightarrow{\ \alpha(X,X)\ } C(X) \otimes C(X) \\ &\xrightarrow{\ \phi\otimes\psi\ } \underline{R} \otimes \underline{R} \xrightarrow{\ \underline{\mu}\ } \underline{R} \end{aligned} \tag{12}$$

of degree $-k-l$, *except* that in order to agree with Hatcher (for example), we must introduce the extra sign $(-1)^{kl}$ to cancel the unwanted sign that (1) introduces here. If ϕ and ψ are cocycles (or chain maps), so is (12). In view of Lemma 9, if we further change ϕ or ψ (or $\alpha(X, X)$) by a chain homotopy, (12) also changes by a chain homotopy, so the resulting cup products in cohomology are well defined.

Closely related is the **cochain cross product** $\phi \times \psi \in C^{k+l}(X \times Y; R)$ of $\phi \in C^k(X; R)$ and $\psi \in C^l(Y; R)$. It corresponds to the homomorphism

$$\phi \times \psi: C(X \times Y) \xrightarrow{\ \alpha(X,Y)\ } C(X) \otimes C(Y) \xrightarrow{\ \phi\otimes\psi\ } \underline{R} \otimes \underline{R} \xrightarrow{\ \underline{\mu}\ } \underline{R} \tag{13}$$

of degree $-k-l$, again with the sign $(-1)^{kl}$. Its properties are similar to, and may be derived from, those of the cup product, and vice versa.

We also have the **chain cross product** $a \times b \in C_{k+l}(X \times Y)$ of $a \in C_k(X)$ and $b \in C_l(Y)$ (taking $R = \mathbb{Z}$ for simplicity). This corresponds to the homomorphism

$$a \times b \colon \underline{\mathbb{Z}} \cong \underline{\mathbb{Z}} \otimes \underline{\mathbb{Z}} \xrightarrow{a \otimes b} C(X) \otimes C(Y) \xrightarrow{\beta(X,Y)} C(X \otimes Y) \tag{14}$$

of degree $k+l$, this time with no extra sign. If a and b are cycles (or chain maps), so is $a \times b$. If we further change a or b by a chain homotopy, $a \times b$ also changes by a chain homotopy, and the resulting cross products in homology are well defined.

The **cap product** $c \cap \phi \in C_{k-l}(X)$ of a chain $c \in C_k(X)$ and cochain $\phi \in C^l(X)$ corresponds to the graded homomorphism

$$\begin{aligned} c \cap \phi \colon \underline{\mathbb{Z}} \xrightarrow{c} C(X) \xrightarrow{\Delta_\#} C(X \times X) &\xrightarrow{\alpha(X,X)} C(X) \otimes C(X) \\ &\xrightarrow{\phi \otimes \mathrm{id}} \underline{\mathbb{Z}} \otimes C(X) \cong C(X) \end{aligned} \tag{15}$$

of degree $k-l$ (with no sign). If c is a cycle (or chain map) and ϕ is a cocycle (or chain map), $c \cap \phi$ is also a chain map. If we further change c or ϕ by a chain homotopy, $c \cap \phi$ also changes by a chain homotopy, and the resulting cap product $\cap \colon H_k(X) \times H^l(X) \to H_{k-l}(X)$ is well defined.

There are also slant products (which are mentioned only briefly by Hatcher in an exercise). There are two. First, given $c \in C_k(X \times Y)$ and $\phi \in C^l(Y)$, their **slant product** $c/\phi \in C_{k-l}(X)$ corresponds to the homomorphism

$$c/\phi \colon \underline{\mathbb{Z}} \xrightarrow{c} C(X \times Y) \xrightarrow{\alpha(X,Y)} C(X) \otimes C(Y) \xrightarrow{\mathrm{id} \otimes \phi} C(X) \otimes \underline{\mathbb{Z}} \cong C(X) \tag{16}$$

of degree $k-l$, with the sign $(-1)^{l(k-l)}$ to cancel out the unwanted sign that (1) introduces. Again, if c and ϕ are chain maps and we change either by a chain homotopy, c/ϕ changes by a chain homotopy, or boundary. Thus it induces the slant product

$$H_k(X \times Y) \times H^l(Y) \longrightarrow H_{k-l}(X). \tag{17}$$

If we take $Y = X$ and $c \in C_k(X)$, comparison with (15) yields (with some care) $\Delta_\# c/\phi = (-1)^{l(k-l)} c \cap \phi$.

Second, given $\phi \in C^k(X \times Y)$ and $c \in C_l(X)$, their **slant product** $\phi/c \in C^{k-l}(X)$ corresponds to the homomorphism

$$\phi/c \colon C(X) \cong C(X) \otimes \underline{\mathbb{Z}} \xrightarrow{\mathrm{id} \otimes c} C(X) \otimes C(Y) \xrightarrow{\beta(X,Y)} C(X \times Y) \xrightarrow{\phi} \underline{\mathbb{Z}} \tag{18}$$

of degree $-k+l$, again with the sign $(-1)^{l(k-l)}$. Once again, if ϕ and c are chain maps, so is ϕ/c, and if we change either by a chain homotopy, ϕ/c changes by a chain homotopy. It therefore induces the slant product

$$H^k(X \times Y) \times H_l(Y) \longrightarrow H^{k-l}(X). \tag{19}$$