General Exponential Functions

For fixed a, we want the exponential function a^x to have at least the properties:

$$\begin{aligned} &\text{(i)} \quad a^{x+y} = a^x a^y; \\ &\text{(ii)} \quad a^0 = 1; \\ &\text{(iii)} \quad a^1 = a; \\ &\text{(iv)} \quad a^x \text{ is continuous in } x. \end{aligned} \tag{1}$$

Many other properties follow easily from these. If we put $y = -x$ in (i), we get $1 = a^0 = a^{x+(-x)} = a^x a^{-x}$, so that

$$a^{-x} = \frac{1}{a^x}, \quad \text{in particular, } a^{-1} = \frac{1}{a^1} = \frac{1}{a}\ . \tag{2}$$

If we iterate (i), we get for any positive integer n

$$a^{nx} = a^{x+x+\ldots+x} = a^x a^x \ldots a^x = (a^x)^n, \tag{3}$$

the nth power of a^x in the ordinary sense. By taking $x = 1$, we see that a^n is the nth power of a as usually understood.

Existence and uniqueness We cannot expect to define a^x for *all* real x and a. We note that axiom (i) gives $a^x = (a^{x/2})^2 \geq 0$. Also, equation (2) requires $a^x \neq 0$, so that we must have $a^x > 0$. Then (iii) gives $a = a^1 > 0$.

THEOREM 4 *Given $a > 0$, there exists a unique function a^x that satisfies the axioms equation* (1). *It satisfies $a^x > 0$ for all x.*

For a rational number $x = p/q$, with p and q integers and $q > 0$, equation (3) gives $(a^x)^q = a^{qx} = a^p$, which determines a^x uniquely. This, with continuity, is enough to determine a^x uniquely for all real x. We defer the proof of existence of a^x.

Applications We make some immediate applications of Theorem 4. The method in each case is to show that a certain function satisfies the axioms equation (1) for a^x for some a, and therefore by the uniqueness in Theorem 4 must coincide with a^x.

Our first application is the elementary fact that $1^x = 1$ for all x. (In detail, define $f(x) = 1$ for all x; then $f(x+y) = 1 = f(x)f(y)$, $f(0) = 1 = f(1)$, and the constant function f is clearly continuous; hence $f(x) = 1^x$.)

Our second application generalizes equation (3).

THEOREM 5 *Given $a > 0$, we have $a^{zx} = (a^z)^x$ for any real z and x.*

Proof Fix a and z and define $g(x) = a^{zx}$ for all x. Then $g(x+y) = a^{z(x+y)} = a^{zx+zy} = a^{zx}a^{zy} = g(x)g(y)$, $g(0) = a^0 = 1$, $g(1) = a^z$, and g is continuous. Thus $g(x)$ satisfies the axioms for $(a^z)^x$ and must coincide with it. □

Our third application shows that a^x is multiplicative in a.

THEOREM 6 *Given $a > 0$ and $b > 0$, we have $(ab)^x = a^x b^x$ for all x.*

Proof We put $h(x) = a^x b^x$ for all x. Then $h(x+y) = a^{x+y} b^{x+y} = a^x a^y b^x b^y = h(x)h(y)$, $h(0) = 1$, $h(1) = a^1 b^1 = ab$, and h is continuous. Thus h satisfies the axioms for $(ab)^x$, and so must be $(ab)^x$. $\square$

Differentiation In order to differentiate a^x, we must take the limit of

$$\frac{a^{x+h} - a^x}{h} = \frac{a^x a^h - a^x}{h} = a^x \frac{a^h - 1}{h}$$

as $h \to 0$. As this limit is not obvious, we give it a name.

DEFINITION 7 Given $a > 0$, we define the *natural logarithm* $\log a$ (or $\ln a$) of a as the limit

$$\log a = \ln a = \lim_{h \to 0} \frac{a^h - 1}{h} \ . \tag{8}$$

When we plug this in, we obtain the answer

$$\frac{d}{dx}(a^x) = a^x \log a \ . \tag{9}$$

THEOREM 10 *We have* $\log a > 0$ *whenever* $a > 1$, $\log 1 = 0$, *and* $\log a < 0$ *whenever* $0 < a < 1$.

Proof As $a^x > 0$ always, equation (9) shows that a^x is an increasing, decreasing, or constant function according as to whether $\log a > 0$, $\log a < 0$, or $\log a = 0$. Comparison of $a^0 = 1$ and $a^1 = a$ shows which case applies. $\square$

The main property of the function $\log x$ follows easily from equation (9).

THEOREM 11 *We have* $\log ab = \log a + \log b$ *for any* $a > 0$ *and* $b > 0$.

Proof When we differentiate in Theorem 6, the product rule gives

$$(ab)^x \log ab = a^x (\log a) b^x + a^x b^x \log b,$$

in which we put $x = 0$. $\square$

Similarly, we can differentiate in Theorem 5.

THEOREM 12 *We have* $\log(a^z) = z \log a$ *for any* $a > 0$ *and any* z.

Proof Differentiation of $a^{zx} = (a^z)^x$ with respect to x gives

$$a^{zx} z \log a = (a^z)^x \log(a^z) \ .$$

Again we put $x = 0$. $\square$