Function Spaces

In standard terminology,

$$\textit{map} = \textit{mapping} = \textit{continuous function.}$$

Given spaces X and Y, denote by Y^X the set of all maps $X \to Y$. (If X is a discrete space with m points and Y has n points, there are n^m maps.) We wish to topologize this set in some reasonable manner. When this is done, it is commonly known as a *function space* (although the term *mapping space* is often used too, and is more accurate).

Example If X is compact and Y is a metric space, it is reasonable to define the *distance* between two maps $g: X \to Y$ and $h: X \to Y$ as

$$D(g,h) = \max_{x \in X} D(g(x), h(x)). \tag{1}$$

Note that $D(g(x), h(x))$ is a continuous real-valued function of x, and because X is compact, the maximum exists. It is readily verified that this defines a metric on Y^X, known as the *metric of uniform convergence.*

Generally, suppose that $f: W \times X \to Y$ is a map, where we impose no conditions on the spaces. For each $w \in W$, we can define a map $f_w: X \to Y$ by $f_w(x) = f(w,x)$; then we construct a function $\hat{f}: W \to Y^X$ by $\hat{f}(w) = f_w$. This suggests the following axiom:

$$\hat{f}: W \to Y^X \textit{ is continuous if and only if } f: W \times X \to Y \textit{ is continuous.} \tag{2}$$

An important special case is when $W = Y^X$ and $\hat{f}$ is the identity map; then f becomes the *evaluation* function $e: Y^X \times X \to Y$, given by $e(g,x) = g(x)$. It is easy to recover f from $\hat{f}$ as

$$f(w,x) = e(\hat{f}(w), x). \tag{3}$$

Uniqueness of such a topology on Y^X is not difficult to see. (If we have two candidates, we take $W = Y^X$ with either topology and $\hat{f}$ as the identity map. Then (2) shows that e is continous with either topology on Y^X, also that $\hat{f}$ is continuous even when the two copies of Y^X have different topologies, in either order. It follows that the two candidate topologies on Y^X coincide.)

The following result is far beyond the scope of this note.

Theorem 4 *In general, no topology on Y^X satisfies axiom* (2) *for all W.*

Moreover, even when such a topology exists, it may not be the most appropriate one for a specific context. Still, we can look for situations where there is a topology that satisfies the axiom.

Theorem 5 *If X is compact and Y is a metric space, the topology on Y^X defined by the metric* (1) *does satisfy axiom* (2).

Proof First, assume that $f\colon W \times X \to Y$ is continuous. We have to show that $\hat{f}$ is continuous. Take $w_0 \in W$ and a disk neighborhood $N(f_{w_0}, p)$ of $f_{w_0} = \hat{f}(w_0)$ in Y^X; we seek a neighborhood M of w_0 in W such that $\hat{f}(M) \subset N(f_{w_0}, p)$. Let P be the open set consisting of all points $(w, x) \in W \times X$ such that $D(f(w_0, x), f(w, x)) < p$; evidently, $w_0 \times X \subset P$. By the tube lemma, since X is assumed compact, there is a neighborhood M of w_0 such that $M \times X \subset P$. Then for any $w \in M$,

$$D(f_{w_0}(x), f_w(x)) = D(f(w_0, x), f(w, x)) < p \qquad \text{for all } x \in X,$$

as required.

For the converse, we see from (3) that we need only show that e is continuous. Take a point $(g, x_0) \in Y^X \times X$ and a disk neighborhood $N(y_0, p)$ of $y_0 = g(x_0)$ in Y. We need neighborhoods $N(g, q)$ of g in Y^X and V of x_0 in X such that $e(N(g, q) \times V) \subset N(y_0, p)$. We take $q = p/2$ and choose V such that $g(V) \subset N(y_0, p/2)$. Then $D(g(x), h(x)) < p/2$ for any $h \in N(g, q)$ and any x; also, $D(y_0, g(x)) < p/2$ if $x \in V$. By the triangle inequality, $D(y_0, e(h, x)) = D(y_0, h(x)) < p$. □

In spite of the previous caution, there *is* a topology on Y^X that works much of the time in practice (and necessarily includes the above example as a special case).

Definition 6 Given a compact subspace $K \subset X$ and an open set $U \subset Y$, denote by $N(K, U)$ the set of all maps $g\colon X \to Y$ that satisfy $g(K) \subset U$. The *compact-open topology* on Y^X is defined as having the collection of all such subsets $N(K, U)$ of Y^X as a subbasis.

Lemma 7 *If $f\colon W \times X \to Y$ is continuous and we give Y^X the compact-open topology, then $\hat{f}\colon W \to Y^X$ is continuous.*

Proof We have only to prove that $\hat{f}^{-1}(N(K, U))$ is open. Take any $w_0 \in W$ such that $\hat{f}(w_0) \in N(K, U)$, i.e. $w_0 \times K \subset f^{-1}(U)$. By the tube lemma, there is a neighborhood M of w_0 in W such that $M \times K \subset f^{-1}(U)$, i.e. $\hat{f}(M) \subset N(K, U)$. □

Lemma 8 *The evaluation function $e\colon Y^X \times X \to Y$ is continuous if we give Y^X the compact-open topology and X is locally compact.*

Proof To show that e is continuous at $(g, x_0) \in Y^X \times X$, take a neighborhood U of $e(g, x_0) = g(x_0)$. We seek neighborhoods N of g and V of x_0 such that $e(N \times V) \subset U$. Now $g^{-1}(U)$ is a neighborhood of x_0. Since X is locally compact, there is a compact subspace $K \subset g^{-1}(U)$ that contains a neighborhood V of x_0. We take $N = N(K, U)$. Since $g(K) \subset U$, $g \in N$. For any $h \in N$, $h(V) \subset h(K) \subset U$, i.e. $e(N \times V) \subset U$. □

Combining these two lemmas with equation (3), we have the main theorem.

Theorem 9 *Let X be locally compact and Y and W any spaces. If we give Y^X the compact-open topology, then $f\colon W \times X \to Y$ is continuous if and only if $\hat{f}\colon W \to Y^X$ is continuous.* □