Derivatives

Given a function f, assume that $f'(a) = m$, where a is fixed. We put

$$\epsilon(h) = \frac{f(a+h) - f(a)}{h} - m \quad \text{for } h \neq 0$$

in order to arrange $\epsilon(h) \to 0$ as $h \to 0$. When we multiply up, we get

$$f(a+h) = f(a) + mh + h\epsilon(h) \tag{1}$$

On the right, we have the constant term $f(a)$, then a constant multiple of h, then an error term which is small compared to h.

Definition 2 The function f is *differentiable* at a, with *derivative* $f'(a) = m$, if and only if equation (1) holds with $\epsilon(h) \to 0$ as $h \to 0$.

In equation (1) we no longer have any division by h, and it remains valid for $h = 0$, whatever value we choose for $\epsilon(0)$. The reasonable choice is $\epsilon(0) = 0$, to make the function ϵ *continuous* at $h = 0$. This form of the definition leads to cleaner proofs of the standard rules.

Theorem 3 *If f is differentiable at a, it is continuous at a.*

Proof We must show that $f(a+h) \to f(a)$ as $h \to 0$. From equation (1) we have

$$f(a+h) \to f(a) + 0 + 0 = f(a) \quad \text{as } h \to 0. \quad \square$$

Constant multiples If we multiply equation (1) by a *constant* c, we get

$$cf(a+h) = cf(a) + cmh + ch\epsilon(h)$$

to which Definition 2 applies immediately. We write $cm = cf'(a)$.

Theorem 4 *Given a constant c, if f is differentiable at a, then so is cf, with derivative $cf'(a)$.* $\square$

Sums and Differences Assume also that $g'(a) = p$, so that like equation (1) we have

$$g(a+h) = g(a) + ph + h\theta(h) \tag{5}$$

with $\theta(h) \to 0$ as $h \to 0$. If we add (or subtract) equations (1) and (5), we see that

$$f(a+h) \pm g(a+h) = f(a) \pm g(a) + (m \pm p)h + h\{\epsilon(h) \pm \theta(h)\}$$

which has the form prescribed by Definition 2 for showing that $f \pm g$ is differentiable.

Theorem 6 *If f and g are differentiable at a, then so is $f \pm g$, with derivative $f'(a) \pm g'(a)$.* $\square$

The Product Rule We can just as well multiply equations (1) and (5). There are nine terms, which we collect as we go,

$$\begin{aligned} f(a+h)g(a+h) =& f(a)g(a) + \{f(a)p + mg(a)\}h \\ & h\{f(a)\theta(h) + mph + mh\theta(h) + \epsilon(h)g(a) + \epsilon(h)ph + h\epsilon(h)\theta(h)\} \end{aligned}$$

Again, Definition 2 applies, and we plug in $p = g'(a)$ and $m = f'(a)$.

Theorem 7 *If f and g are differentiable at a, then so is fg, with derivative*

$$f'(a)g(a) + f(a)g'(a) \quad \square$$

The Chain Rule Suppose that $u = g(x)$ and $y = f(u)$, so that $y = f(u) = f(g(x))$. Assume first that $g'(a) = p$, so that we have by equation (5)

$$\begin{aligned} g(a+h) &= g(a) + ph + h\theta(h) \\ &= b + k \end{aligned}$$

where we write $b = g(a)$ and $k = h(p + \theta(h))$ in preparation for the next step, which is to expand $f(g(a+h)) = f(b+k)$. We note that $k \to 0$ as $h \to 0$.

Assume that $f'(b) = q$, so that like equation (1),

$$\begin{aligned} f(g(a+h)) = f(b+k) &= f(b) + qk + k\epsilon(k) \\ &= f(b) + qph + h\{q\theta(h) + p\epsilon(k) + \theta(h)\epsilon(k)\} \end{aligned}$$

where $\epsilon(k) \to 0$ as $k \to 0$ and we substituted for k in some places. Once again, Definition 2 applies, and the derivative of $f \circ g$ is qp. (It is possible to have $k = 0$ exactly, but this is not a problem if we define $\epsilon(0) = 0$ as suggested, to make ϵ continous.) We plug in $q = f'(b) = f'(g(a))$ and $p = g'(a)$.

Theorem 8 *If g is differentiable at a and f is differentiable at $g(a)$, then the composite $f \circ g$ is differentiable at a, with derivative $f'(g(a))\, g'(a)$.* $\square$

Reciprocals We apply Theorem 8 with $f(u) = 1/u$, for which $f'(u) = -1/u^2$.

Theorem 9 *If f is differentiable at a, then so is $1/f$, with derivative $\dfrac{-f'(a)}{f(a)^2}$ (provided that $f(a) \neq 0$).* $\square$

The Quotient Rule

Theorem 10 *If u and v are differentiable at a, then so is $\dfrac{u}{v}$, with derivative*

$$\frac{v(a)u'(a) - u(a)v'(a)}{v(a)^2}$$

provided $v(a) \neq 0$.

Proof We write $u/v = u.(1/v)$ and apply the product rule Theorem 7 and Theorem 9, to get

$$u'(a)\frac{1}{v(a)} + u(a)\frac{-v'(a)}{v(a)^2}$$

which can be rearranged as stated. $\square$