Derivatives and Differentials

References are to Salas/Hille/Etgen's *Calculus*, 8th Edition (Wiley, 1999)

We wish to differentiate a scalar-valued function $f(\mathbf{r})$ of a vector variable $\mathbf{r}$ at the point $P_0 = \mathbf{r}_0$. For simplicity, we assume $\mathbf{r}$ is 2-dimensional and write $\mathbf{r}_0 = (x_0, y_0)$; however, everything extends without difficulty (except notationally) to n dimensions.

Differentials We want our basic definitions to be coordinate-free so as to ensure some geometric content. We reduce to a *one*-dimensional problem by restricting attention to the values of f along the general line through $\mathbf{r}_0$ with direction vector $\mathbf{h} = h_1\mathbf{i} + h_2\,\mathbf{j}$, which may parametrized by $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{h}$, by setting

$$g(t) = f(\mathbf{r}(t)) = f(\mathbf{r}_0 + t\mathbf{h}); \tag{1}$$

this function is defined for all small t (if $\mathbf{r}_0$ is an interior point of the domain of f).

Definition 2 Given the point $\mathbf{r}_0$ and the vector $\mathbf{h}$, we define the *differential* df of f by

$$df(\mathbf{r}_0, \mathbf{h}) = g'(0) = \lim_{t\to 0} \frac{g(t) - g(0)}{t} = \lim_{t\to 0} \frac{f(\mathbf{r}_0 + t\mathbf{h}) - f(\mathbf{r}_0)}{t}, \tag{3}$$

provided that this limit exists.

Note that here $\mathbf{h}$ can be *any* vector; it does not have to be small and it does not have to be a unit vector.

Directional derivatives If $\mathbf{h}$ is a *unit* vector, $df(\mathbf{r}_0, \mathbf{h})$ is also called (see Defn. 15.2.2) the *directional derivative of f at the point $\mathbf{r}_0$ in the direction $\mathbf{h}$*, and is sometimes written $f'_{\mathbf{h}}(\mathbf{r}_0)$ or $D_{\mathbf{h}}f(\mathbf{r}_0)$. (We shall not use either notation.) But if $\mathbf{h}$ is not a unit vector (and not zero), we can replace $\mathbf{h}$ by the unit vector $(1/\|\mathbf{h}\|)\,\mathbf{h}$ and call $df(\mathbf{r}_0, (1/\|\mathbf{h}\|)\,\mathbf{h})$ the directional derivative of f *in the direction of* $\mathbf{h}$ (see the Remark on p. 882). This usage in the book is confusing and is not recommended.

Partial derivatives For purposes of computation, we are particularly interested in the two coordinate directions, $\mathbf{i}$ and $\mathbf{j}$. It is convenient to write $f(\mathbf{r}) = f(x, y)$ here.

Definition 4 (cf. Defn. 14.4.1) The *partial derivative* $f_x(\mathbf{r}_0)$ or $\dfrac{\partial f}{\partial x}(\mathbf{r}_0)$ *of f with respect to x at the point* $\mathbf{r}_0 = (x_0, y_0)$ is the directional derivative

$$f_x(\mathbf{r}_0) = f_x(x_0, y_0) = \frac{\partial f}{\partial x}(\mathbf{r}_0) = df(\mathbf{r}_0, \mathbf{i}) = \lim_{t\to 0} \frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t}. \tag{5}$$

Similarly, the *partial derivative of f with respect to y at* $\mathbf{r}_0$ is

$$f_y(\mathbf{r}_0) = f_y(x_0, y_0) = \frac{\partial f}{\partial y}(\mathbf{r}_0) = df(\mathbf{r}_0, \mathbf{j}) = \lim_{t\to 0} \frac{f(x_0, y_0 + t) - f(x_0, y_0)}{t}. \tag{6}$$

If $u = f(\mathbf{r})$, one often writes u_x for f_x and $\dfrac{\partial u}{\partial x}$ for $\dfrac{\partial f}{\partial x}$.

Linearity It is easy to see that $df(\mathbf{r}_0, k\mathbf{h}) = k\,df(\mathbf{r}_0, \mathbf{h})$ for any constant k. (Changing $\mathbf{h}$ to $k\mathbf{h}$ replaces $g(t)$ in equation (1) by $g(kt)$.) However, the values of f along different lines through $\mathbf{r}_0$ are quite independent, apart from the obvious requirement $g(0) = f(\mathbf{r}_0)$. This makes it quite easy to manufacture all kinds of bizarre behavior of df; see the book for some examples. (The existence of df does not guarantee that f is continuous at $\mathbf{r}_0$, see Example 2 on p. 860, or even that f is bounded near $\mathbf{r}_0$.)

Nevertheless, such pathology rarely occurs in practice. Instead of moving from P_0 to P $(= \mathbf{r}_0 + \mathbf{h} = (x_0 + h_1, y_0 + h_2))$ directly, we can go by way of Q $(= (x_0, y_0 + h_2))$; from P_0 to Q, we change only the y-coordinate, and from Q to P we change only the x-coordinate. One often writes Δx for h_1 and Δy for h_2 here. If the partial derivatives of f exist at all points near $\mathbf{r}_0$, we can estimate $f(P) - f(P_0) = (f(P) - f(Q)) + (f(Q) - f(P_0))$ by using the one-dimensional Mean-Value Theorem, in the form

$$g(t+h) - g(t) = g'(t + \theta h)\,h \tag{7}$$

for some θ satisfying $0 < \theta < 1$, assuming that g' exists.

THEOREM 8 (cf. Thm. 15.1.3) *Assume that f_x and f_y are continuous at $\mathbf{r}_0$ (and therefore exist at all points near $\mathbf{r}_0$). Then f is differentiable at $\mathbf{r}_0$,*

$$f(\mathbf{r}_0 + \mathbf{h}) = f(\mathbf{r}_0) + f_x(\mathbf{r}_0)\,h_1 + f_y(\mathbf{r}_0)\,h_2 + o(\mathbf{h}) \qquad \text{as } \mathbf{h} \to \mathbf{0}. \tag{9}$$

Proof See the book, p. 878. For the notation $o(\mathbf{h})$, see p. 872. □

Under these mild hypotheses, we easily compute the differential df.

THEOREM 10 *If f_x and f_y are continuous at $\mathbf{r}_0$, the differential df is given by*

$$df(\mathbf{r}_0, \mathbf{h}) = \nabla f(\mathbf{r}_0) \bullet \mathbf{h} = f_x(\mathbf{r}_0)\,h_1 + f_y(\mathbf{r}_0)\,h_2 = \frac{\partial f}{\partial x}(\mathbf{r}_0)\,h_1 + \frac{\partial f}{\partial y}(\mathbf{r}_0)\,h_2. \tag{11}$$

Proof If we use equation (9) (with $\mathbf{h}$ replaced by $t\mathbf{h}$), equation (1) becomes

$$g(t) = f(\mathbf{r}_0 + t\mathbf{h}) = f(\mathbf{r}_0) + f_x(\mathbf{r}_0)\,th_1 + f_y(\mathbf{r}_0)\,th_2 + o(t) \qquad \text{as } t \to 0.$$

Now we can read off $g'(0)$ as required. The term $o(t)$ contributes nothing. □

Approximation If we plug equation (11) back into equation (9), we obtain

$$f(\mathbf{r}_0 + \mathbf{h}) = f(\mathbf{r}_0) + df(\mathbf{r}_0, \mathbf{h}) + o(\mathbf{h}) \qquad \text{as } \mathbf{h} \to \mathbf{0}, \tag{12}$$

which shows how the differential df gives a useful approximation to $f(\mathbf{r}_0 + \mathbf{h})$.

The coordinate differentials The coordinate x can itself be considered a function of $\mathbf{r}$, and its differential is readily computed from equation (11), or directly from equation (3), as $dx(\mathbf{r}_0, \mathbf{h}) = 1h_1 + 0h_2 = h_1$. Similarly, $dy(\mathbf{r}_0, \mathbf{h}) = h_2$. This allows us to rewrite equation (11) as the equation of differentials

$$df(\mathbf{r}_0, \mathbf{h}) = f_x(\mathbf{r}_0)\,dx(\mathbf{r}_0, \mathbf{h}) + f_y(\mathbf{r}_0)\,dy(\mathbf{r}_0, \mathbf{h}). \tag{13}$$

If we suppress all the evaluations and switch to the other notation, this takes the traditional and memorable form (cf. (15.7.4))

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{14}$$

So df/dx is *not* the same as $\partial f/\partial x$, unless dy happens to be zero.