Evaluation of Double Integrals

The following Fubini-type theorem is fundamental to the evaluation of any Riemann double integral. Yet I have found it stated in this form only in the advanced text by Apostol (and even there, only with a complicated proof of a special case). It is in fact quite easy, and is a result in pure integration (or measure) theory, with no mention of continuity; it is obvious for step functions. We sketch a direct proof below. Everything here generalizes (with a little care) to three dimensions, or even n dimensions.

THEOREM 1 *Assume that the function $f(x, y)$ is integrable over the rectangle R defined by $a \leq x \leq b$, $c \leq y \leq d$ in the xy-plane.*

(a) *We have*

$$\iint_R f(x, y)\,dx\,dy = \int_c^d F(y)\,dy = \int_c^d \left\{ \int_a^b f(x, y)\,dx \right\} dy,$$

provided that the inner integral

$$F(y) = \int_a^b f(x, y)\,dx$$

exists for all y ($c \leq y \leq d$). In particular, the function F is integrable.

(b) *We have*

$$\iint_R f(x, y)\,dx\,dy = \int_a^b G(x)\,dx = \int_a^b \left\{ \int_c^d f(x, y)\,dy \right\} dx,$$

provided that the inner integral

$$G(x) = \int_c^d f(x, y)\,dy$$

exists for all x ($a \leq x \leq b$). In particular, the function G is integrable.

Examples show that the integrability hypothesis on f is essential. For this result to be useful, we need a supply of integrable functions.

THEOREM 2 *Suppose that the function $f(x, y)$ is bounded on the rectangle R, and is continuous except on a set E with area zero. Then f is integrable over R.*

We do not give the proof, which is quite advanced even if we assume that f is continuous everywhere. And this special case fails to cover many essential applications, such as the following corollary.

COROLLARY 3 *Let Ω be a bounded region in the plane. If the set of boundary points of Ω has area zero, then the area of Ω is defined.*

Proof We apply Theorem 2 to the characteristic function χ_Ω of Ω. □

In view of this corollary, Theorem 2 does look rather circular. To make it useful, we need a supply of sets that are known to have area zero. Of course, the union of any two (or finitely many) sets of area zero also has area zero.

THEOREM 4 *The following sets have area zero:*

(i) *The graph $y = f(x)$ of an integrable function on the interval $a \le x \le b$;*

(ii) *The graph $x = g(y)$ of an integrable function on the interval $c \le y \le d$;*

(iii) *The image curve of a function $\mathbf{r}(t)$ that is continuously differentiable on the interval $a \le t \le b$.*

We omit the proofs. Parts (i) and (ii) are again pure integration theory, but somewhat messier, and are closely related to the following result.

THEOREM 5 *Suppose that $f(x)$ is integrable on the interval $a \le x \le b$, and that $f(x) \ge 0$ for all x. Then the region Ω under the graph of f, defined by $0 \le y \le f(x)$ and $a \le x \le b$, has area $\int_a^b f(x)dx$.*

Once we know that the area exists, it is readily computed by Theorem 1.

Sketch proof of Theorem 1(a) We have to show that F is integrable, and that its integral is $I = \int\int_R f(x,y)dxdy$.

So take any Riemann sum

$$\sum_{j=1}^{n} F(y_j^*)(y_j - y_{j-1}) \tag{6}$$

for the integral of F, where $c = y_0 < y_1 < \ldots < y_n = d$ is a partition (or subdivision) of $[c, d]$ and $y_{j-1} \le y_j^* \le y_j$ for each j. We also choose a partition $a = x_0 < x_1 < \ldots < x_m = b$ of $[a, b]$, thus dividing the rectangle R into little rectangles R_{ij} as usual.

For each j, we choose points $P_{ij} \in R_{ij}$ having y_j^* as the y-coordinate; then

$$\sum_{i=1}^{m} f(P_{ij})(x_i - x_{i-1}) \tag{7}$$

is a Riemann sum for the integral

$$\int_a^b f(x, y_j^*)dx = F(y_j^*),$$

and will therefore be close to $F(y_j^*)$ if the x-partition is fine enough. This holds for all j simultaneously, if the x-partition is fine enough to handle all the integrals $F(y_j^*)$; this works because we are considering here only finitely many integrals.

Now multiply (7) by $y_j - y_{j-1}$ and add. Then

$$\sum_{j=1}^{n}\sum_{i=1}^{m} f(P_{ij})(x_i - x_{i-1})(y_j - y_{j-1}) \tag{8}$$

will be close to (6). But (8) is a Riemann sum for the double integral I, and is therefore close to I if the partitions of $[a, b]$ and $[c, d]$ are fine enough. So (6) is close to (8) which is close to I, as required. □

As it stands, this is not a formal proof, but can easily be made so. One has to insert δ and ϵ in all the right places, and check that the various choices are made in a legitimate order. The method also works well if the definite integral is defined in terms of upper and lower sums instead of Riemann sums.