The Tangent Vector to a Curve

Let C be a space curve parametrized by the differentiable vector-valued function $\mathbf{r}(t)$. At the point P_0 $(= \mathbf{r}(t_0))$ of C, we have the *derivative* (or *velocity*) vector

$$\mathbf{v}(t_0) = \mathbf{r}'(t_0) = \lim_{h\to 0} \frac{\mathbf{r}(t_0+h) - \mathbf{r}(t_0)}{h} = \lim_{h\to 0} \frac{\overrightarrow{P_0P}}{h}, \tag{1}$$

where we write P for the point $\mathbf{r}(t_0+h)$ on C. We give a direct geometric interpretation of this vector, *assuming it is not zero.* It follows from equation (1) that for small $h \neq 0$, we have $P \neq P_0$, so that $\overrightarrow{P_0P}$ is a nonzero vector.

THEOREM 2 *Assume that* $\mathbf{r}'(t_0) \neq \mathbf{0}$. *Let* $\theta(h)$ *be the angle between the vectors* $\overrightarrow{P_0P}$ *and* $\mathbf{r}'(t_0)$ *(which is defined for sufficiently small* $h \neq 0$*). Then:*

(a) $\lim_{h\to 0+} \theta(h) = 0$;

(b) $\lim_{h\to 0-} \theta(h) = \pi$; *or equivalently, the angle between* $\overrightarrow{P_0P}$ *and* $-\mathbf{r}'(t_0)$ *tends to* 0 *as* $h \to 0-$.

Proof The standard angle formula gives

$$\cos\theta(h) = \frac{\overrightarrow{P_0P} \bullet \mathbf{r}'(t_0)}{\left\|\overrightarrow{P_0P}\right\| \|\mathbf{r}'(t_0)\|} = \frac{[\mathbf{r}(t_0+h) - \mathbf{r}(t_0)] \bullet \mathbf{r}'(t_0)}{\|\mathbf{r}(t_0+h) - \mathbf{r}(t_0)\| \|\mathbf{r}'(t_0)\|}$$

We cannot take the limit directly, because we get $0/0$. However, if we first divide numerator and denominator by h and assume $h > 0$, we can apply equation (1) directly (using the continuity of norms and dot products) to get

$$\cos\theta(h) = \frac{\dfrac{\mathbf{r}(t_0+h) - \mathbf{r}(t_0)}{h} \bullet \mathbf{r}'(t_0)}{\left\|\dfrac{\mathbf{r}(t_0+h) - \mathbf{r}(t_0)}{h}\right\| \|\mathbf{r}'(t_0)\|} \longrightarrow \frac{\mathbf{r}'(t_0) \bullet \mathbf{r}'(t_0)}{\|\mathbf{r}'(t_0)\| \|\mathbf{r}'(t_0)\|} = 1 \tag{3}$$

as $h \to 0+$. Finally, we apply the continuous function $\cos^{-1}$ to deduce that

$$\theta(h) = \cos^{-1}(\cos\theta(h)) \longrightarrow \cos^{-1}(1) = 0 \qquad \text{as } h \to 0+,$$

as required.

If $h < 0$, equation (3) is off by a sign, and we get $\theta \to \cos^{-1}(-1) = \pi$ instead. □

Geometrically, the vector $\mathbf{r}'(t_0)$ is *tangent* to the curve C at P_0. This leads to the following definition.

DEFINITION 4 The *tangent line to* C *at* P_0 is the line through P_0 in the direction of the vector $\mathbf{r}'(t_0)$.

Thus its parametric equation (with parameter u) is (see (13.3.2))

$$\mathbf{R}(u) = \mathbf{r}(t_0) + u\mathbf{r}'(t_0). \tag{5}$$