The Cone Operator in Singular Homology

This note reorganizes some material in Hatcher's "Algebraic Topology", §2.1.

We introduce the *cone operator*, as a tool for the homotopy and excision axioms.

Linear chains As a first step, we study the following situation. Suppose Z is a convex subspace of $\mathbb{R}^q$. Given any $n+1$ points $v_0, v_1, \ldots, v_n$ in Z (*not* necessarily linearly independent), we have the linear map

$$\lambda = [v_0, v_1, \ldots, v_n]\colon \Delta^n \longrightarrow Z \tag{1}$$

from the standard n-simplex $\Delta^n \subset \mathbb{R}^{n+1}$ with vertices e_i (for $0 \le i \le n$), defined by $\lambda(e_i) = v_i$ for each i. Its image in Z is thus

$$\lambda(\Delta^n) = \left\{ \sum_{i=0}^{n} t_i v_i : t_i \ge 0 \text{ for all } i, \quad \sum_{i=0}^{n} t_i = 1 \right\}, \tag{2}$$

which may or may not be a geometric simplex.

Such maps λ generate the subgroup $LC_n(Z) \subset C_n(Z)$ of *linear n-chains* in Z. As n varies, the groups $LC_n(Z)$ clearly form a sub-chain complex $LC_*(Z)$ of the singular chain complex $C_*(Z)$. The space Z, being convex, is contractible.

Proposition 3 *$H_0(LC_*(Z)) \cong \mathbb{Z}$, and $H_n(LC_*(Z)) = 0$ for $n \ne 0$.*

Suppose $Z' \subset \mathbb{R}^s$ is another convex subspace, and that $A\colon \mathbb{R}^q \to \mathbb{R}^s$ is a linear (or affine) map that satisfies $A(Z) \subset Z'$. It is clear that the chain map $A_\#\colon C_*(Z) \to C_*(Z')$ restricts to a chain map

$$A_\#\colon LC_*(Z) \longrightarrow LC_*(Z'). \tag{4}$$

The cone operator This is suggested by the fact that the cone on a simplex is another simplex.

Definition 5 Given a point $v \in Z$, we define the *cone operator*

$$C_v\colon LC_n(Z) \longrightarrow LC_{n+1}(Z)$$

as the homomorphism defined on each linear simplex (1) by

$$C_v\lambda = C_v[v_0, v_1, v_2, \ldots, v_n] = [v, v_0, v_1, v_2, \ldots, v_n] \in LC_{n+1}(Z). \tag{6}$$

Proposition 7 *With $A\colon Z \to Z'$ as in* (4), *we have*

$$A_\# \circ C_v = C_{Av} \circ A_\#\colon LC_*(Z) \longrightarrow LC_*(Z'). \quad \square$$

Let us compute the boundary.

$$\begin{aligned} \partial C_v\lambda &= [v_0, v_1, \ldots, v_n] - \sum_{i=0}^{n} (-1)^i [v, v_0, v_1, \ldots, \widehat{v_i}, \ldots, v_n] \\ &= \lambda - C_v\partial\lambda. \end{aligned}$$

This calculation is valid only for $n > 0$. For $\lambda = [v_0]$ we have instead

$$\partial C_v\lambda = \partial[v, v_0] = [v_0] - [v] = \lambda - [v].$$

By taking linear combinations, for any chain $c \in LC_n(Z)$ we may write

$$\partial C_v c = \begin{cases} c - C_v \partial c & \text{if } n > 0; \\ c - (\epsilon c)[v] & \text{if } n = 0; \end{cases} \tag{8}$$

where we introduce the *augmentation* homomorphism $\epsilon\colon LC_0(Z) = C_0(Z) \to \mathbb{Z}$ given by $\epsilon[z] = 1$ for all $z \in Z$.

Proof of Proposition 3 It is not necessary to determine the n-cycles or n-boundaries in $LC_*(Z)$. If $c \in LC_n(Z)$ is a cycle, where $n > 0$, (8) shows that $c = \partial C_v c$ and is thus a boundary.

Any chain $c \in LC_0(Z)$ is a 0-cycle, and by (8) is homologous to some multiple $(\epsilon c)[v]$ of $[v]$. It follows that $H_0(LC_*(Z)) \cong \mathbb{Z}$, generated by $[v]$. □

Chain homotopies The usefulness of equation (8) suggests a definition.

Definition 9 Given two chain maps $f, g\colon C \to C'$ between chain complexes C and C', a *chain homotopy from f to g* is a family of homomorphisms $s_n\colon C_n \to C'_{n+1}$ that satisfy

$$\partial s_n c + s_{n-1} \partial c = g_n c - f_n c \qquad \text{for all } n \text{ and all } c \in C_n. \tag{10}$$

We say the two chain maps are *chain homotopic* and write $f \simeq g\colon C \to C'$.

It is easy to see that being chain homotopic is an equivalence relation. The argument of Proposition 3 generalizes immediately.

Proposition 11 *If $f, g\colon C \to C'$ are chain homotopic chain maps, we have $f_* = g_*\colon H_n(C) \to H_n(C')$.*

Proof If $c \in C_n$ is a cycle, (10) reduces to $g_n c - f_n c = \partial s_n c$, which shows that the cycles $f_n c$ and $g_n c$ are homologous. □

The prism operator, on linear chains Let Z be a convex subspace of $\mathbb{R}^q$, as before. We compare the two chain maps $j_{0\#}, j_{1,\#}\colon LC_*(Z) \to LC_*(Z \times I)$ induced by the maps j_0 and j_1 defined by $j_r(z) = (z, r)$.

Given a linear n-simplex $\lambda = [u_0, u_1, \ldots, u_n]$ in Z, we write the two simplices $j_0 \circ \lambda$ and $j_1 \circ \lambda$ as $[v_0, v_1, \ldots, v_n]$ and $[w_0, w_1, \ldots, w_n]$ respectively; they give the two ends of a prism $\Delta^n \times I$ in $Z \times I$ (when the u_i are linearly independent). Geometrically, although the prism is not a simplex, it *can* be regarded as a cone with vertex v_0 on the union of the top face $\Delta^n \times 1$ and $F_0 \times I$, where F_0 denotes the face of Δ^n opposite u_0 (the image of $\eta_0\colon \Delta^{n-1} \subset \Delta^n$). (See the pictures on page 112.) This suggests the following algebraic definition.

Definition 12 We define the *prism operator* homomorphism

$$P'_n\colon LC_n(Z) \longrightarrow LC_{n+1}(Z \times I)$$

on any linear n-simplex $\lambda = [u_0, u_1, \ldots, u_n]$ as above by

$$P'_n \lambda = C_{v_0}(j_{1\#}\lambda - P'_{n-1} d_0 \lambda) \in LC_{n+1}(Z \times I) \qquad \text{for } n \geq 0, \tag{13}$$

using induction on n, where $v_0 = (u_0, 0) \in Z \times I$. Of course, $P'_n = 0$ for all $n < 0$.

Note that the vertex v_0 of the cone varies as λ varies.

It is clear that P'_n commutes with linear maps, in the following sense.

PROPOSITION 14 *Let $A: Z \to Z'$ be a linear (or affine) map as in (4). Then*

$$P'_n \circ A_\# = (A \times \mathbf{1}_I)_\# \circ P'_n: LC_n(Z) \longrightarrow LC_{n+1}(Z' \times I). \quad \square$$

LEMMA 15 *Let Z be a convex subspace of $\mathbb{R}^q$. Then for all n,*

$$\partial \circ P'_n + P'_{n-1} \circ \partial = j_{1\#} - j_{0\#}: LC_n(Z) \longrightarrow LC_n(Z \times I). \tag{16}$$

In words, the homomorphisms P'_n form a chain homotopy P' from $j_{0\#}$ to $j_{1\#}$.

Proof For $n = 0$, $P'_0\lambda = [v_0, w_0]$ and

$$\partial P'_0\lambda = \partial[v_0, w_0] = [w_0] - [v_0] = j_{1\#}\lambda - j_{0\#}\lambda.$$

For $n > 0$, we proceed inductively:

$$\begin{aligned}
\partial P'_n\lambda &= \partial C_{v_0} j_{1\#}\lambda - \partial C_{v_0} P'_{n-1} d_0\lambda \quad \text{by (13)} \\
&= j_{1\#}\lambda - C_{v_0}\partial j_{1\#}\lambda - P'_{n-1}d_0\lambda + C_{v_0}\partial P'_{n-1}d_0\lambda \quad \text{by (8)} \\
&= j_{1\#}\lambda - C_{v_0} j_{1\#}\partial\lambda - P'_{n-1}d_0\lambda - C_{v_0}P'_{n-2}\partial d_0\lambda + C_{v_0}j_{1\#}d_0\lambda - C_{v_0}j_{0\#}d_0\lambda,
\end{aligned}$$

using (16) for $n-1$ and the induction hypothesis. Here, the second and fifth terms combine as

$$-\sum_{i=1}^{n} (-1)^i C_{v_0} j_{1\#} d_i\lambda. \tag{17}$$

In the fourth term, we expand

$$\partial d_0\lambda = \sum_{k=0}^{n-1} (-1)^k d_k d_0 \lambda = \sum_{k=0}^{n-1} (-1)^k d_0 d_{k+1}\lambda,$$

so that the fourth term becomes

$$-C_{v_0}P'_{n-2}\partial d_0\lambda = \sum_{k=0}^{n-1} (-1)^{k+1} C_{v_0} P'_{n-2} d_0 d_{k+1}\lambda. \tag{18}$$

The last term reduces to one we want,

$$C_{v_0} j_{0\#}[u_1, \ldots, u_n] = C_{v_0}[v_1, \ldots, v_n] = [v_0, v_1, \ldots, v_n] = j_{0\#}\lambda.$$

Meanwhile, again using (13), for P'_{n-1},

$$\begin{aligned}
P'_{n-1}\partial\lambda &= P'_{n-1}d_0\lambda + \sum_{i=1}^{n} (-1)^i P'_{n-1} d_i\lambda \\
&= P'_{n-1}d_0\lambda + \sum_{i=1}^{n} (-1)^i C_{v_0} j_{1\#} d_i\lambda - \sum_{i=1}^{n} (-1)^i C_{v_0} P'_{n-2} d_0 d_i \lambda,
\end{aligned}$$

where we note that for $i > 0$, the leading vertex of $d_i\lambda$ is still u_0. When we add $\partial P'_n\lambda$ to this, the first term cancels out, the second term cancels (17), and the third cancels (18), if we replace i by $k+1$. $\square$

The prism operator, on general chains We extend the prism operator P' to general spaces and chains by working in the space $\Delta^n \times I$ rather than X or Y. We focus attention on the element $\delta_n \in LC_n(\Delta^n) \subset C_n(\Delta^n)$, which denotes the identity map $\mathbf{1}\colon \Delta^n \to \Delta^n$, considered as a singular n-simplex of the space Δ^n. Then for any singular n-simplex $\sigma\colon \Delta^n \to X$, the chain map $\sigma_\#\colon C_*(\Delta^n) \to C_*(X)$ induces $\sigma_\# \delta_n = \sigma$.

DEFINITION 19 Given a homotopy $f_t\colon X \to Y$, hence a map $F\colon X \times I \to Y$, we define the prism operator homomorphism $P_n\colon C_n(X) \to C_{n+1}(Y)$ on the singular n-simplex $\sigma\colon \Delta^n \to X$, a generator of $C_n(X)$, by

$$P_n\sigma = F_\#(\sigma \times \mathbf{1}_I)_\# P'_n \delta_n. \tag{20}$$

Remark We note that if X is a convex subspace of $\mathbb{R}^q$, σ a linear singular n-simplex of X, and $Y = X \times I$ with F the identity map of $X \times I$, equation (20) reduces to

$$P_n\sigma = (\sigma \times \mathbf{1}_I)_\# P'_n \delta_n = P'_n \sigma_\# \delta_n = P'_n \sigma,$$

with the help of Proposition 14. So P_n *does* extend P'_n.

THEOREM 21 *Given a homotopy $f_t\colon X \to Y$, the resulting prism operator satisfies*

$$\partial \circ P_n + P_{n-1} \circ \partial = f_{1\#} - f_{0\#}\colon C_n(X) \longrightarrow C_n(Y). \tag{22}$$

COROLLARY 23 *Then $f_* = g_*\colon H_n(X) \to H_n(Y)$.* □

Proof of Theorem 21 We evaluate each term of (22) on a general singular n-simplex $\sigma\colon \Delta^n \to X$:

$$\begin{aligned}
\partial P_n\sigma &= \partial F_\#(\sigma \times \mathbf{1}_I)_\# P'_n \delta_n = F_\#(\sigma \times \mathbf{1}_I)_\# \partial P'_n \delta_n;\\
P_{n-1}\partial\sigma &= \sum_{i=0}^{n} (-1)^i P_{n-1}(\sigma \circ \eta_i) = \sum_{i=0}^{n} (-1)^i F_\#(\sigma \times \mathbf{1}_I)_\# (\eta_i \times \mathbf{1}_I)_\# P'_{n-1}\delta_{n-1}\\
&= F_\#(\sigma \times \mathbf{1}_I)_\# P'_{n-1} \left\{ \sum_{i=0}^{n} (-1)^i \eta_{i\#}\delta_{n-1} \right\} \quad \text{by Proposition 14}\\
&= F_\#(\sigma \times \mathbf{1}_I)_\# P'_{n-1} \partial\delta_n;\\
f_{1\#}\sigma &= f_1 \circ \sigma = F \circ (\sigma \times \mathbf{1}_I) \circ j_1 = F_\#(\sigma \times \mathbf{1}_I)_\# j_{1\#}\delta_n;\\
f_{0\#}\sigma &= F_\#(\sigma \times \mathbf{1}_I)_\# j_{0\#}\delta_n, \quad \text{similarly.}
\end{aligned}$$

Then (22) follows from (16). □

Barycentric subdivision, linear case We define the barycentric subdivision first on linear simplices, to produce a chain map $S'_n\colon LC_n(Z) \to LC_n(Z)$. Geometrically, we proceed by induction; once the faces of a linear n-simplex λ as in (1) have been subdivided, we join everything to the *barycenter* $b(\lambda) = \frac{1}{n+1}\sum_{i=0}^{n} v_i$ of λ.

Algebraically, we begin the induction with $S'_0 = \mathbf{1}$, and continue with

$$S'_n\lambda = C_{b(\lambda)} S'_{n-1}\partial\lambda \in LC_n(Z) \qquad \text{for } n > 0. \tag{24}$$

(Of course, $S'_n = 0$ for $n < 0$.) We must check that S' is a chain map. For $n = 1$ we have, using (8),

$$\partial S'_1\lambda = \partial C_{b(\lambda)} S'_0 \partial\lambda = S'_0\partial\lambda - \epsilon(S'_0\partial\lambda)[b(\lambda)] = S'_0\partial\lambda,$$

since $\epsilon(S'_0\partial\lambda) = \epsilon(\partial\lambda) = 1 - 1 = 0$. For $n \ge 2$ we compute, again using (8),

$$\partial S'_n\lambda = \partial C_{b(\lambda)} S'_{n-1}\partial\lambda = S'_{n-1}\partial\lambda - C_{b(\lambda)}\partial S'_{n-1}\partial\lambda = S'_{n-1}\partial\lambda,$$

since by induction $\partial S'_{n-1}\partial\lambda = S'_{n-2}\partial\partial\lambda = 0$.

We also need a chain homotopy T' between S' and the identity chain map $\mathbf{1}$, i.e. homomorphisms $T'_n\colon LC_n(Z) \to LC_{n+1}(Z)$ that satisfy

$$\partial \circ T'_n + T'_{n-1} \circ \partial = \mathbf{1} - S'_n\colon LC_n(Z) \longrightarrow LC_n(Z) \qquad \text{for all } n. \tag{25}$$

Geometrically (see the picture on page 122), we subdivide the lower face $\Delta^n \times 0$ of the prism $\Delta^n \times I$, leave the upper face $\Delta^n \times 1$ alone, and join the barycenter $(b(\lambda), 0)$ of the lower face to $\Delta^n \times 1$ and the already subdivided vertical faces $\partial\Delta^n \times I$ of the prism, then project everything to Z.

Algebraically, we begin the induction with $T'_0 = 0$. (Hatcher uses $T'_0[v_0] = [v_0, v_0]$, which works equally well. Again, $T'_n = 0$ for $n < 0$.) We continue with

$$T'_n\lambda = C_{b(\lambda)}(\lambda - T'_{n-1}\partial\lambda) \in LC_{n+1}(Z) \qquad \text{for } n > 0. \tag{26}$$

Then equation (25) is trivial for $n = 0$. We verify it for $n > 0$ by induction, by evaluating it on λ, with the help of (8),

$$\partial T'_n\lambda = \partial C_{b(\lambda)}(\lambda - T'_{n-1}\partial\lambda) = \lambda - T'_{n-1}\partial\lambda - C_{b(\lambda)}\partial\lambda + C_{b(\lambda)}\partial T'_{n-1}\partial\lambda.$$

We use (25) for $n - 1$ to expand the last term,

$$C_{b(\lambda)}\partial T'_{n-1}\partial\lambda = -C_{b(\lambda)}T'_{n-2}\partial\partial\lambda + C_{b(\lambda)}\partial\lambda - C_{b(\lambda)}S'_{n-1}\partial\lambda.$$

The first term on the right vanishes, the second cancels the unwanted term in $\partial T'_n\lambda$, and the third is $S'_n\lambda$ by definition.

The following property of S'_n and T'_n is immediate.

Proposition 27 *With $A\colon Z \to Z'$ as in (4), the chain map $A_\#\colon LC_n(Z) \to LC_n(Z')$ commutes with S'_n and T'_n: $S'_n \circ A_\# = A_\# \circ S'_n$ and $T'_n \circ A_\# = A_\# \circ T'_n$.* □

Barycentric subdivision, general case As with the prism operator, we extend the definition of S'_n and T'_n to a general singular n-simplex $\sigma\colon \Delta^n \to X$ of any space X by using the chain map $\sigma_\#\colon LC_n(\Delta^n) \subset C_n(\Delta^n) \to C_n(X)$, where $\delta_n \in LC_n(\Delta^n)$ denotes as before the singular simplex $\mathbf{1}\colon \Delta^n \to \Delta^n$. For all n, we define the homomorphisms $S_n\colon C_n(X) \to C_n(X)$ and $T_n\colon C_n(X) \to C_{n+1}(X)$ on each generator σ by

$$S_n\sigma = \sigma_\# S'_n\delta_n; \qquad T_n\sigma = \sigma_\# T'_n\delta_n. \tag{28}$$

If X happens to be a convex subspace of a real vector space and σ is a linear (or affine) n-simplex, these reduce by Proposition 27 to $S'_n\sigma$ and $T'_n\sigma$.

To verify that S is a chain map, we expand, using the fact that $\sigma_\#$ and S' are chain maps,

$$\partial S_n\sigma = \partial\sigma_\# S'_n\delta_n = \sigma_\#\partial S'_n\delta_n = \sigma_\# S'_{n-1}\partial\delta_n,$$

$$S_{n-1}\partial\sigma = \sum_{i=0}^{n}(-1)^i S_{n-1}(\sigma \circ \eta_i) = \sum_{i=0}^{n}(-1)^i \sigma_\#\eta_{i\#} S'_{n-1}\delta_{n-1}.$$

These agree, since

$$\partial\delta_n = \sum_{i=0}^{n}(-1)^i\delta_n \circ \eta_i = \sum_{i=0}^{n}(-1)^i\eta_i = \sum_{i=0}^{n}(-1)^i\eta_{i\#}\delta_{n-1} \tag{29}$$

and $\eta_{i\#}$ commutes with S' by Proposition 27.

A similar proof shows that T satisfies

$$\partial T_n\sigma + T_{n-1}\partial\sigma = \sigma - S_n\sigma \colon C_n(X) \longrightarrow C_n(X), \tag{30}$$

so is a chain homotopy from S to $\mathbf{1}$. We expand each term,

$$\begin{aligned}
\partial T_n\sigma &= \partial\sigma_\# T'_n\delta_n = \sigma_\#\partial T'_n\delta_n = \sigma_\# T'_n\partial\delta_n,\\
T_{n-1}\partial\sigma &= \sum_{i=0}^{n}(-1)^i T_{n-1}(\sigma\circ\eta_i) = \sum_{i=0}^{n}(-1)^i\sigma_\#\eta_{i\#}T'_{n-1}\delta_{n-1},\\
\sigma &= \sigma_\#\delta_n,\\
S_n\sigma &= \sigma_\# S'_n\delta_n.
\end{aligned}$$

Since T'_{n-1} commutes with $\eta_{i\#}$ by Proposition 27, (30) now follows from (25).

We also need to know that S'_n subdivides any linear simplex λ as in (1) into simplices that really are smaller. There are $(n+1)!$ simplices in the barycentric subdivision of λ; one of these is

$$\lambda' = \left[v_0, \frac{v_0+v_1}{2}, \frac{v_0+v_1+v_2}{3}, \ldots, \frac{v_0+v_1+\ldots+v_n}{n+1}\right].$$

All the others are obtained by reordering the vertices of λ. In terms of equation (2), its image in $Z \subset \mathbb{R}^q$ is

$$\lambda'(\Delta^n) = \left\{\sum_{i=0}^{n} t_i v_i : t_0 \ge t_1 \ge \ldots \ge t_n \ge 0, \quad \sum_{i=0}^{n} t_i = 1\right\} \tag{31}$$

Because t_0 is the largest and $\sum_{i=0}^n t_i = 1$, we must have $t_0 \ge \frac{1}{n+1}$.

Lemma 32 *We have* $\operatorname{diam}\lambda'(\Delta^n) \le \frac{n}{n+1}\operatorname{diam}\lambda(\Delta^n)$.

Proof Consider the linear simplex $\lambda'' = [v_0, v''_1, v''_2, \ldots, v''_n]$, where $v''_i = \frac{1}{n+1}v_0 + \frac{n}{n+1}v_i$ for $1 \le i \le n$. This simplex is geometrically similar to λ, except shrunk in all directions by the factor $\frac{n}{n+1}$, keeping v_0 fixed; therefore $\operatorname{diam}\lambda''(\Delta^n) = \frac{n}{n+1}\operatorname{diam}\lambda(\Delta^n)$.

Now the image $\lambda''(\Delta^n)$ consists of all the points of $\lambda(\Delta^n)$ with $t_0 \ge \frac{1}{n+1}$, and therefore contains $\lambda'(\Delta^n)$. □