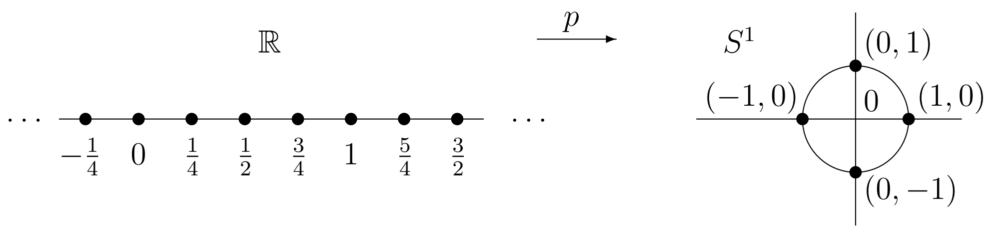
A Covering Space of the Circle

This note fills in details in Hatcher, §1.1, page 30.

We take S^1 to be the unit circle in $\mathbb{C}$, the complex numbers, or equivalently, in the plane $\mathbb{R}^2$. Consider the map $p\colon \mathbb{R} \to S^1$ given by $p(t) = e^{2\pi it} \in \mathbb{C}$, or equivalently (as in Hatcher) $p(t) = (\cos 2\pi t, \sin 2\pi t) \in \mathbb{R}^2$. Informally, it wraps the real line around the circle. We note that p is periodic, $p(t+1) = p(t)$ for all t, so that we need discuss in detail only values of t that lie in (or near) the unit interval, $[0,1]$.



Definition 1 Given a map $p\colon E \to B$, a *local section* of p is any map $s\colon U \to E$, where U is open in B, such that $p \circ s\colon U \to B$ is the inclusion, i.e. $s(x) \in p^{-1}(x)$ for all $x \in U$. The map s is a (*global*) *section* if $U = B$.

Definition 2 Given a map $p\colon E \to B$, the open set $U \subset B$ is *evenly covered* by p if $p^{-1}(U)$ is the disjoint union of open sets $\widetilde{U}_\lambda$ in E with local sections $s_\lambda\colon U \to \widetilde{U}_\lambda$ that are bijections (and hence homeomorphisms, with inverses $p|\widetilde{U}_\lambda$).

Theorem 3 *The map $p\colon \mathbb{R} \to S^1$ is a covering map, i.e. there exists a covering of S^1 by open sets U_i, each of which is evenly covered by p.*

Remark The map p admits no global section, as will be clear later.

Remark The terminology is unfortunately confusing; the word *covering* has two quite different meanings here. In French, there are two separate words: *recouvrement* for an open covering of a space, and *revêtement* for a covering space.

The proof of Theorem 3 is essentially Calculus I.

Proof We use *four* open sets, U_0, U_1, U_2 and U_3, described in terms of $\mathbb{R}^2$ rather than $\mathbb{C}$.

Take $U_0 = \{(x,y) \in S^1 : x > 0\}$. As $x^2 + y^2 = 1$, y determines x on U_0 by $x = \sqrt{1-y^2}$. We have the local section $s_{0,0}\colon U_0 \to \widetilde{U}_{0,0} = (-\frac14, \frac14)$ given by $s_{0,0}(x,y) = \frac{1}{2\pi}\sin^{-1} y$; as y increases from -1 to 1, $\sin^{-1} y$ increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and $s_{0,0}(x,y)$ increases from $-\frac14$ to $\frac14$. More generally, we have for any integer n the local section $s_{0,n}\colon U_0 \to \widetilde{U}_{0,n} = (n-\frac14, n+\frac14)$ given by $s_{0,n}(x,y) = n + s_{0,0}(x,y)$. For other points of $\mathbb{R}$, e.g. $n + \frac14 \le t \le n + \frac34$, $\cos 2\pi t = \cos 2\pi(t-n) \le 0$ and $p(t) \notin U_0$. So U_0 is evenly covered by the intervals $(n-\frac14, n+\frac14)$.

Take $U_1 = \{(x,y) \in S^1 : y > 0\}$. Here, x determines $y = \sqrt{1-x^2}$. We have the local section $s_{1,0}\colon U_1 \to \widetilde{U}_{1,0} = (0, \frac12)$ given by $s_{1,0}(x,y) = \frac{1}{2\pi}\cos^{-1} x$; as x decreases from 1 to -1, $\cos^{-1} x$ increases from 0 to π and $s_{1,0}(x,y)$ increases from 0 to $\frac12$. As with U_0, U_1 is evenly covered by the intervals $\widetilde{U}_{1,n} = (n, n+\frac12)$.

Take $U_2 = \{(x,y) \in S^1 : x < 0\}$. Here, y determines $x = -\sqrt{1-y^2}$. We have the local section $s_{2,0}\colon U_2 \to \widetilde{U}_{2,0} = (\frac{1}{4}, \frac{3}{4})$ given by $s_{2,0}(x,y) = \frac{1}{2} + \frac{1}{2\pi}\sin^{-1}(-y)$; as y decreases from 1 to -1, $-y$ increases from -1 to 1, $\sin^{-1}(-y)$ increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and $s_{2,0}(x,y)$ increases from $\frac{1}{4}$ to $\frac{3}{4}$. Then U_2 is evenly covered by the intervals $\widetilde{U}_{2,n} = (n+\frac{1}{4}, n+\frac{3}{4})$.

Take $U_3 = \{(x,y) \in S^1 : y < 0\}$. Here, x determines $y = -\sqrt{1-x^2}$. We have the local section $s_{3,0}\colon U_3 \to \widetilde{U}_{3,0} = (\frac{1}{2}, 1)$ given by $s_{3,0}(x,y) = \frac{1}{2} + \frac{1}{2\pi}\cos^{-1}(-x)$; as x increases from -1 to 1, $-x$ decreases from 1 to -1, $\cos^{-1}(-x)$ increases from 0 to π and $s_{3,0}(x,y)$ increases from $\frac{1}{2}$ to 1. Then U_3 is evenly covered by the intervals $\widetilde{U}_{3,n} = (n+\frac{1}{2}, n+1)$. □