Attaching a Cell

Compare Proposition 1.26 in Hatcher.

We determine the effect on the fundamental group of attaching one cell to a general space X. We assume X is path-connected, with basepoint x_0.

Attaching a 2-cell Attach a 2-cell to X using the attaching map $\phi\colon S^1 \to X$ to form the space Y, so that we have the pushout square of spaces

$$\begin{array}{ccc} D^2 & \xrightarrow{\Phi} & Y \\ \cup\uparrow i & & \cup\uparrow j \\ S^1 & \xrightarrow{\phi} & X \end{array} \tag{1}$$

Choose a basepoint $d_1 \in S^1$ and put $x_1 = f(d_1) \in X$. Write $\omega_1\colon I \to S^1$ for the standard loop in S^1, so that $\pi_1(S^1, d_1) = \mathbb{Z}$, generated by $[\omega_1]$. Since ω_1 is nullhomotopic in D^2, $\phi \circ \omega_1$ is nullhomotopic in Y. Take a path γ in X from x_0 to x_1.

THEOREM 2 *The homomorphism $j_*\colon \pi_1(X, x_0) \to \pi_1(Y, x_0)$ induced by the inclusion $j\colon X \subset Y$ is surjective, and induces the isomorphism of groups $\pi_1(X, x_0)/N \cong \pi_1(Y, x_0)$, where N is the smallest normal subgroup of $\pi_1(X, x_0)$ that contains the element $\beta_\gamma[\phi \circ \omega_1] = \beta_\gamma(\phi_*[\omega_1])$.*

LEMMA 3 *In the commutative diagram*

$$\begin{array}{ccccc} \pi_1(D^2, d_1) & \xrightarrow{\Phi_*} & \pi_1(Y, x_1) & \underset{\beta_\gamma}{\xrightarrow{\cong}} & \pi_1(Y, x_0) \\ \uparrow i_* & & \uparrow j_* & & \uparrow j_* \\ \pi_1(S^1, d_1) & \xrightarrow{\phi_*} & \pi_1(X, x_1) & \underset{\beta_\gamma}{\xrightarrow{\cong}} & \pi_1(X, x_0) \end{array} \tag{4}$$

the left square, hence the whole rectangle, are pushout squares of groups.

Proof of Theorem 2 Given homomorphisms $\psi_1\colon \pi_1(D^2, d_1) \to G$ and $\psi_2\colon \pi_1(X, x_0) \to G$ such that $\psi_1 \circ i_* = \psi_2 \circ \beta_\gamma \circ \phi_*$, where G is any group, we seek a unique homomorphism $\psi\colon \pi_1(Y, x_0) \to G$ such that $\psi \circ \beta_\gamma \circ \Phi_* = \psi_1$ and $\psi \circ j_* = \psi_2$. As D^2 is contractible, the first requirement is trivial and $\operatorname{Ker} \psi_2$ must contain the element $\beta_\gamma \phi_*[\omega_1]$, and hence the whole of N. We have enough to identify $\pi_1(Y, x_0)$ with $\pi_1(X, x_0)/N$. □

Proof of Lemma 3 The given pushout square (1) does not lend itself to direct application of van Kampen's Theorem. The key idea is to *attach a collar* to X.

Define the collar $V = \{x \in D^2 : \|x\| > 1/2\}$ of S^1 in D^2; it contains S^1 as a deformation retract. We attach V to X using the same attaching map $\phi\colon S^1 \to X$ to produce the subspace $X \cup_\phi V$ of Y. Now we expand diagram (1) to the following diagram,

$$\begin{array}{ccccc} D^2 - S^1 & \xrightarrow{\subset} & D^2 & \xrightarrow{\Phi} & Y \\ \cup\uparrow & & \cup\uparrow & & \cup\uparrow \\ V - S^1 & \xrightarrow{\subset} & V & \xrightarrow{\Phi|V} & X \cup_\phi V \\ & & \cup\uparrow i & & \cup\uparrow j \\ & & S^1 & \xrightarrow{\phi} & X \end{array} \tag{5}$$

which contains *five* pushout squares, by the stacking properties of pushout squares. (The top left square is a pushout of open subspaces.)

We need a new basepoint $d_2 = 3d_1/4 \in V \subset D^2$ that does not lie in S^1. Now we *can* apply van Kampen, taking $A_1 = Y - X$ and $A_2 = X \cup_\phi V$ with basepoint $y_2 = \Phi(d_2)$, to obtain the pushout square of groups and homomorphisms induced by inclusion

$$\begin{array}{ccc} \pi_1(A_1, y_2) & \longrightarrow & \pi_1(Y, y_2) \\ \uparrow & & \uparrow \\ \pi_1(A_1 \cap A_2, y_2) & \longrightarrow & \pi_1(A_2, y_2) \end{array}$$

The top left group is trivial, as Φ maps $D^2 - S^1$ homeomorphically to A_1 and $D^2 - S^1$ is contractible. Further, Φ maps the subspace $V - S^1$ homeomorphically to $A_1 \cap A_2$, so we may replace the bottom homomorphism by

$$(\Phi|(V - S^1))_*\colon \pi_1(V - S^1, d_2) \longrightarrow \pi_1(A_2, y_2).$$

As $V - S^1 \subset V$ is an obvious homotopy equivalence, we may in turn replace this by $(\Phi|V)_*\colon \pi_1(V, d_2) \to \pi_1(A_2, y_2)$.

Finally, we reduce to the desired homomorphism ϕ_* by the commutative diagram

$$\begin{array}{ccc} \pi_1(V, d_2) & \xrightarrow{(\Phi|V)_*} & \pi_1(A_2, y_2) \\ \cong \downarrow \beta_h & & \cong \downarrow \beta_{\Phi\circ h} \\ \pi_1(V, d_1) & \xrightarrow{(\Phi|V)_*} & \pi_1(A_2, x_1) \\ \cong \uparrow i_* & & \cong \uparrow j_* \\ \pi_1(S^1, d_1) & \xrightarrow{\phi_*} & \pi_1(X, x_1) \end{array}$$

where h denotes a path in D^2 from d_1 to d_2. Since S^1 is a deformation retract of V, X is a deformation retract of A_2 and we have the two lower isomorphisms. □

Attaching an n-cell We now consider attaching an n-cell to X to form Y, where $n > 2$, using an attaching map $\phi\colon S^{n-1} \to X$. Then diagram (4) becomes

$$\begin{array}{ccccc} \pi_1(D^n, d_1) & \xrightarrow{\Phi_*} & \pi_1(Y, x_1) & \underset{\beta_\gamma}{\xrightarrow{\cong}} & \pi_1(Y, x_0) \\ \uparrow i_* & & \uparrow j_* & & \uparrow j_* \\ \pi_1(S^{n-1}, d_1) & \xrightarrow{\phi_*} & \pi_1(X, x_1) & \underset{\beta_\gamma}{\xrightarrow{\cong}} & \pi_1(X, x_0) \end{array}$$

However, since $\pi_1(S^{n-1}, d_1)$ is now trivial, i_* and hence j_* are isomorphisms.

PROPOSITION 6 *Form the space Y from X by attaching an n-cell using an attaching map $\phi\colon S^{n-1} \to X$, where $n > 2$. Then the inclusion map $j\colon X \subset Y$ induces an isomorphism $j_*\colon \pi_1(X, x_0) \cong \pi_1(Y, x_0)$.* □