Adams Operations on Group Representations

This note expands the topic of Adams operations, which receive only a brief mention in §7 of Chapter II in Bröcker–tom Dieck.

We work in the complex representation ring $R(G)$ of a compact Lie group G. We write $[V] \in R(G)$ for the class of a representation V.

Exterior powers Given a representation V, we have the *exterior powers* $\Lambda^k V$ for $k \geq 0$; $\Lambda^0 V \cong \mathbb{C}$, $\Lambda^1 V \cong V$, and $\Lambda^k V = 0$ for $k > \dim V$. There is a canonical decomposition

$$\Lambda^k(V \oplus W) \cong \bigoplus_{j=0}^{k} (\Lambda^j V \otimes \Lambda^{k-j} W). \tag{1}$$

This formula makes it convenient to bundle all the exterior powers together.

Definition 2 Given a representation V, we define a polynomial over the ring $R(G)$ in the formal indeterminate t,

$$\lambda_t(V) = \sum_k [\Lambda^k V] t^k \qquad \text{in } R(G)[t]. \tag{3}$$

This allows us to rewrite equation (1) as

$$\lambda_t(V \oplus W) = \lambda_t(V)\lambda_t(W). \tag{4}$$

There is an obvious way to extend λ_t to a function $\lambda_t \colon R(G) \to R(G)[[t]]$, the ring of formal power series in t over $R(G)$. Any element $x \in R(G)$ can be written in the form $x = [V] - [W]$; we set $\lambda_t(x) = \lambda_t(V)\lambda_t(W)^{-1}$, taking advantage of the fact that the polynomial $\lambda_t(W)$ begins with the term 1 and so can be inverted to yield a power series. Further, equation (4) guarantees that $\lambda_t(x)$ is independent of the choices of V and W and also satisfies the identity

$$\lambda_t(x + y) = \lambda_t(x)\lambda_t(y), \tag{5}$$

which extends equation (4) to virtual representations. We extract the coefficients from $\lambda_t(x)$ and define operations $\lambda^k \colon R(G) \to R(G)$ by the identity

$$\lambda_t(x) = \sum_{k=0}^{\infty} (\lambda^k x) t^k. \tag{6}$$

These extend the exterior powers on representations, $\lambda^k[V] = [\Lambda^k V]$; also $\lambda^0 x = 1$ and $\lambda^1 x = x$ for all x.

However, the structure of $\Lambda^k(V \otimes W)$ and $\Lambda^m(\Lambda^k V)$ remains obscure and difficult to handle. Nevertheless, in $R(G)$ it is possible to expand $\lambda^k(xy)$ and $\lambda^m(\lambda^k x)$ as polynomials in the $\lambda^r x$ and $\lambda^r y$. The two simplest non-trivial examples are

$$\lambda^2(xy) = x^2(\lambda^2 y) + (\lambda^2 x)y^2 - 2(\lambda^2 x)(\lambda^2 y), \qquad \lambda^2(\lambda^2 x) = x(\lambda^3 x) - \lambda^4 x;$$

even these are far from obvious. Note that there are negative terms. Below, we give one way to compute such polynomials. The ring $R(G)$, equipped with the operations

λ^k which satisfy the appropriate polynomial identities, becomes what is known as a λ-*ring*. We do not pursue this approach any further.

Adams operations The obvious way to make equation (5) more useful is to take logarithms formally, using the standard logarithmic series

$$\log(1+Z) = Z - \frac{Z^2}{2} + \frac{Z^3}{3} - \frac{Z^4}{4} + \dots$$

to obtain the additive formula

$$\log \lambda_t(x+y) = \log \lambda_t(x) + \log \lambda_t(y) \qquad \text{in } (R(G) \times \mathbb{Q})[[t]],$$

where we now work over $R(G) \otimes \mathbb{Q}$ because we introduced fractions; no real damage is done, as $R(G)$ is a free abelian group with no torsion. Still, we can easily remove the fractions by differentiating (i. e. performing *logarithmic differentiation*) to obtain

$$\frac{\frac{d}{dt}\lambda_t(x+y)}{\lambda_t(x+y)} = \frac{\frac{d}{dt}\lambda_t(x)}{\lambda_t(x)} + \frac{\frac{d}{dt}\lambda_t(y)}{\lambda_t(y)}.$$

Of course, we can bypass all mention of logarithms and fractions and obtain this directly by differentiating equation (5) by the product rule and then dividing by (5).

We need to normalize. If $x = [V]$, where $\dim V = 1$, we have $\lambda_t(x) = 1 + xt$ and

$$\frac{\frac{d}{dt}\lambda_t(x)}{\lambda_t(x)} = \frac{x}{1+xt} = x - x^2t + x^3t^2 - x^4t^3 + \dots$$

Definition 7 We define the additive *Adams operations* $\Psi^k\colon R(G) \to R(G)$ for positive integers k by the identity

$$\frac{\frac{d}{dt}\lambda_t(x)}{\lambda_t(x)} = \Psi^1 x - (\Psi^2 x)t + (\Psi^3 x)t^2 - (\Psi^4 x)t^3 + \dots \tag{8}$$

This equation allows us to compute Ψ^k in terms of the operators λ^r by picking out the coefficient of $\pm t^{k-1}$ in

$$\{x + 2(\lambda^2 x)t + 3(\lambda^3 x)t^2 + \dots\}\{1 - (xt + (\lambda^2 x)t^2 + \dots) + (xt + (\lambda^2 x)t^2 + \dots)^2 - \dots\}.$$

Obviously, $\Psi^1 x = x$. The next three are

$$\begin{aligned} \Psi^2 x &= x^2 - 2\lambda^2 x; \\ \Psi^3 x &= x^3 - 3x(\lambda^2 x) + 3\lambda^3 x; \\ \Psi^4 x &= x^4 + 4x(\lambda^3 x) + 2(\lambda^2 x)^2 - 4x^2(\lambda^2 x) - 4\lambda^4 x. \end{aligned} \tag{9}$$

Conversely, one can solve these for $\lambda^k x$ in terms of the $\Psi^r x$, at the cost of introducing some fractions. When working solely with the Ψ^k, some divisibility properties are lost, such as that $\Psi^2 x + x^2$ is divisible by 2 in $R(G)$.

THEOREM 10 *For each integer k, there exists a unique operation $\Psi^k\colon R(G) \to R(G)$ for compact Lie groups G that satisfies the axioms*

$$\begin{aligned} &\text{(i)}\ \Psi^k\colon R(G)\to R(G) \text{ is additive, } \Psi^k(x+y)=\Psi^k x+\Psi^k y;\\ &\text{(ii)}\ \Psi^k \text{ is natural: if } \rho\colon H\to G \text{ is a homomorphism of Lie groups, } \Psi^k \text{ commutes with } \rho^*\colon R(G)\to R(H),\ \rho^*\Psi^k x=\Psi^k\rho^* x;\\ &\text{(iii)}\ \text{If } x=[V], \text{ where } \dim V=1,\ \Psi^k x=x^k. \end{aligned} \tag{11}$$

So far, we have constructed Ψ^k, for positive k only, to satisfy (i) and (iii); since ρ^* preserves exterior powers, it also preserves Ψ^k and we have (ii).

The rather uninteresting operation Ψ^0 is in effect the augmentation $\rho^*\colon R(G) \to R(\{e\}) \cong \mathbb{Z}$ induced by the inclusion $\{e\} \subset G$; if $x = [V]-[W]$, $\Psi^0 x = \dim V - \dim W$.

We take Ψ^{-1} to be complex conjugation, which clearly satisfies the axioms (11), once we note that in (iii), $[\overline{V}][V] = [V^*][V] = [V^* \otimes V] = 1$, where V^* denotes the dual of V and the evaluation map $V^* \otimes V \to \mathbb{C}$ here is an equivariant isomorphism. (One can also verify (iii) using characters). More generally, we define $\Psi^{-k} x = \Psi^k \bar{x}$.

One noteworthy property of Ψ^k is its effect on characters.

PROPOSITION 12 *We have $\chi_{\Psi^k x}(g) = \chi_x(g^k)$ for all $g \in G$, $x \in R(G)$ and $k \in \mathbb{Z}$.*

Proof Since Ψ^k is additive, it is sufficient to check for $x = [V]$, V a representation. Fix $g \in G$ and let H be the closed subgroup of G generated by g; this is a compact abelian Lie group with inclusion $\rho\colon H \subset G$. Then $\rho^* V$ decomposes into one-dimensional irreducible representations of H,

$$\rho^* V = W_1 \oplus W_2 \oplus \ldots \oplus W_n.$$

For each j, we have

$$\chi_{\Psi^k W_j}(g) = \chi_{W_j^{\otimes k}}(g) = \chi_{W_j}(g)^k = \chi_{W_j}(g^k),$$

since $\chi_{W_j}\colon H \to S^1$ is a homomorphism of groups. Adding, we have

$$\chi_{\Psi^k V}(g) = \chi_{\rho^*\Psi^k V}(g) = \chi_{\Psi^k \rho^* V}(g) = \chi_{\rho^* V}(g^k) = \chi_V(g^k). \quad \square$$

Proof of Theorem 10 We have only uniqueness to prove. The element $\Psi^k x \in R(G)$ is determined by its character, which was computed in the proof of Proposition 12 using only the axioms (11). □

THEOREM 13 *For any integers k and l, we have:*

(a) *$\Psi^k\colon R(G) \to R(G)$ is a ring homomorphism, $\Psi^k(xy) = (\Psi^k x)(\Psi^k y)$;*
(b) *$\Psi^k \circ \Psi^l = \Psi^{kl}$.*

Proof For (a), it is enough to check that the characters agree. By Proposition 12,

$$\chi_{\Psi^k(xy)}(g) = \chi_{xy}(g^k) = \chi_x(g^k)\chi_y(g^k) = \chi_{\Psi^k x}(g)\chi_{\Psi^k y}(g).$$

In (b), it is clear that $\Psi^k \circ \Psi^l$ satisfies the axioms (11) for Ψ^{kl} and therefore is Ψ^{kl}. (This is also easy to see using characters.) □

Remark We can use this result to obtain formulae for $\lambda^k(xy)$ and $\lambda^m(\lambda^k x)$. We express each λ^k in terms of the Ψ^r by solving equations (9), apply Theorem 13, then use (9) to convert each Ψ^r back. The fractions that this process introduces all disappear at the end (though this method fails to make it apparent why this should happen).

Symmetric powers Instead of exterior powers, one can use symmetric powers throughout. We have a similar formula to (1),

$$S^k(V \oplus W) \cong \bigoplus_{j=0}^{k} (S^j V \otimes S_{k-j} W),$$

can define the series $\sigma_t(V) = \sum_k [S^k V] t^k \in R(G)[[t]]$, and extend it to $x = [V] - [W] \in R(G)$ by $\sigma_t(x) = \sigma_t(V)\sigma_t(W)^{-1}$. Again, the identity $\sigma_t(x+y) = \sigma_t(x)\sigma_t(y)$ leads to consideration of the series $(\frac{d}{dt}\sigma_t(x))\sigma_t(x)^{-1}$. For $x = [V]$, where $\dim V = 1$, we find

$$\frac{\frac{d}{dt}\sigma_t(x)}{\sigma_t(x)} = \frac{x + 2x^2 t + 3x^3 t^2 + \dots}{1 + xt + x^2 t^2 + \dots} = x + x^2 t + x^3 t^2 + \dots$$

(with no signs). We therefore define the operators Ψ^k for general x by

$$\frac{\frac{d}{dt}\sigma_t(x)}{\sigma_t(x)} = \Psi^1 x + (\Psi^2 x)t + (\Psi^3 x)t^2 + (\Psi^4 x)t^3 + \dots$$

These new operators Ψ^k clearly satisfy the axioms (11) and therefore by Theorem 10 agree with the previous definition. A closer study reveals that $\sigma_t(x) = \lambda_{-t}(x)^{-1}$.

Real representations In exactly the same way, one can define Adams operations Ψ^k on the *real* representation ring $RO(G)$, with the obvious simplification that $\Psi^{-k} = \Psi^k$. However, axiom (11)(iii) no longer suffices ($\Psi^2 \neq \Psi^0$, even though $x^2 = 1$) and Theorem 10 fails as stated.

Instead, we observe that complexification $RO(G) \to R(G)$ is injective and preserves exterior powers and hence Ψ^k, as well as characters. Thus the real Ψ^k may be uniquely determined as the restriction to $RO(G)$ of the complex $\Psi^k \colon R(G) \to R(G)$. It follows that Proposition 12 holds for real representations too.