110:615 **algebraic topology I**, Fall 2016

Topology is the newest branch of mathematics. It originated around the turn of the twentieth century in response to Cantor, though its roots go back to Euler; it stands between algebra and analysis, and has had profound effects on both.

Since the 1950s topology has been at the cutting edge of mathematical research: its techniques have revolutionized algebraic geometry, number theory, physics (eg of condensed matter, not to mention string theory), as well as important parts (elliptic PDEs) of analysis. Since about 2000 it has been a significant source of ideas for the analysis of large structured data sets, and lately (via HoTT = higher order type theory) it has led to a rethinking of the foundations of logic and the philosophy of mathematics.

Topologists classify space for a living: for example, the three-dimensional space of politicians, as a subset of the five-dimensional manifold of real human beings. Knots and braids provide another class of interesting spaces, as does our eleven-dimensional (or so they would have us believe) own physical Universe. Phylogenetic trees in evolutionary theory are another class of examples just beginning to be studied.

615 is an introduction to algebraic topology as a way of thinking: not only in terms of its techniques, but as an opportunity to introduce a rich supply of concrete, and perhaps surprising, examples.

Draft schedule for 110.615 **classical algebraic topology**

It is a truth universally acknowledged, that there is no really satisfactory introductory algebraic topology textbook. This course attempts to verify this by providing a new example, cherrypicked from the best parts of several quite good standard choices.

A draft schedule follows. The course starts with a review of background material from geometry, which will be used as test examples throughout the course. The semester ends with the Poincaré duality theorem for manifolds; it is intended to lead into a second course centered around the model categorical approach to homotopy theory and homological algebra.

Please contact me ⟨`jack@math.jhu.edu`⟩ if you are interested, or have any questions. A rough draft for the course lectures is attached.

Part ∅ **Introductory material**

Part I **Background from geometry**

Part II **The Euler characteristic and its categorification**

— Week VI: (10 October)

3.8 examples: invariance of dimension, degree of a map, the orientation sheaf, the class of a submanifold, attaching a cell, p 59

Part III **Complexes and chains**

4.1 (abstract) simplicial complexes, eg Rips complexes, partition posets; simplicial chains, p 68
4.2 geometric realization, p 72
4.3 barycentric subdivision, p 74

— Week VII: (17 October)

4.4 products, p 76
4.5 simplicial sets; the classifying space of a category, BG and homotopy quotients, p 77

BASIC HOMOLOGICAL ALGEBRA AND VERIFICATION OF THE AXIOMS

5.1 chain complexes, chain homomorphisms, and chain homotopies, p 81

— Week VIII: (24 October)

5.2 singular homology; the homotopy axiom, p 88
5.3 locality of the singular complex, p 90
5.4 the snake lemma and the boundary homomorphism, p 96: MOVIETIME!

$$\texttt{https}://\texttt{www.youtube.com/watch?v}=\texttt{etbcKWEKnvg}$$

— Week IX (31 October)

Part IV **Back to geometry!**

THE STABLE HOMOTOPY CATEGORY OF FINITE CELL COMPLEXES

6.1 cell complexes; the homotopy type of a cell complex, cellular chains, p 98
6.2 uniqueness of homology. Statement (not proof!) of theorems of Whitehead and Kan, p 102
6.3 sketch of the stable homotopy category, versus the homotopy category of chain complexes. Naive definition of naive spectra; statement (not proof) of Brown's representability theorem, p 105

— Week X (7 November)

COHOMOLOGY

7.1 Definition, axioms for the algebra and module structures; the Alexander-Whitney map, p 112
7.2 examples, p 117
7.2 cap products; the Eilenberg-Zilber map; the Künneth theorem foreshadowed, p 120

— Week XI (14 November)

POINCARÉ DUALITY

8.1 introduction, p 129
8.1 The orientation class, p 125
8.2 proof of the theorem, p 128

— Week XII (28 November)

8.4 applications: Intersection theory and Lefschetz' theorem. The Pontryagin-Thom collapse map, the Thom isomorphism theorem; bivariant functors, p 131

— Week XIII (5 December): **Margin for error!**

[**Appendices:** (to appear?)

On π_1: van Kampen, Hurewicz; Reidemeister moves and braid groups; Wirtinger's presentation of $\pi_1(S^3 - k)$; skein relations and the Alexander polynomial; covering spaces, eg of surfaces and configuration spaces; twisted coefficients; Chern classes, eg of line bundles; elliptic curves and

$$1 \to \mathbb{Z} \to \mathrm{Br}_3 \to \mathrm{Sl}_2(\mathbb{Z}) \to 1 \ .$$

deRham cohomology: Poincaré's lemma; the Hodge operator and duality; Maxwell's equations]

References:

R Ghrist, **Elementary applied topology** (2014)

M Greenberg, **Lectures on algebraic topology** (1967)

J Rotman, **An introduction to algebraic topology** (1988)

A Hatcher, **Algebraic topology** (2002)

Politicians' uniquely simple personalities

The complexity of human personality has been reduced to five dimensions, based on factor analyses of judgements of personality traits[1]. Many researchers agree that a five-factor model of personality captures the essential features of all traits that are used to describe personality: energy/extroversion; agreeableness/friendliness; conscientiousness; emotional stability against neuroticism; and intellect/openness to experience[2–4]. But we show here that this common, standard set of five factors does not hold for judgements of famous political figures.

We found that, when people judge the personality traits of politicians, they use only two or three factors. Personality factors that are normally independent — such as energy and openness — were highly correlated in a more simplified view of personality.

Political candidates gain intense media exposure over an extended period of self-promotion designed to portray them as trustworthy experts with many admirable personality traits[5]. Such public exposure is intended to lead to clearly articulated perceptions rather than stereotypical evaluations by the electorate[6].

The nature of campaign information is unique as a basis for forming impressions of personality, as it is packaged by supporters and opponents as pros and cons (favourable or condemning) designed to simplify the ultimately dichotomous decision of how to vote. The selective mental processing and filtering by the electorate of the mass of discrepant input about political candidates must in the end justify each person's one vote: be it for or against. Therefore, we predicted that personality judgements about political candidates would likewise be constricted to involve a limited number of factors rather than the usual five.

We first studied the personality judgements of a sample of 2,088 Italian adults, of diverse ages, education and political views. Leading party politicians were evaluated by 1,257 respondents, and another 831 evaluated their own personalities and those of several celebrities. Judgements were made from a list of 25 adjectives that are markers of the five-factor model. Each adjective (for example, enterprising, reliable, truthful) was rated on how characteristic it was of each target on a seven-point scale, and those ratings were factor-analysed[7]. The analysis reduces the scores to a minimal number of correlated groups of traits within factors that are independent of each other.

Ratings were made of two Italian political candidates (Silvio Berlusconi and Roman Prodi), an international celebrity (skiing hero Alberto Tomba) and a famous Italian television personality (Pippo Baudo).

Table 1 reveals three clear results: (1) respondents' personality portraits of themselves require the five-factor solution, as found in earlier research; (2) personality judgements of national celebrities also require five factors; but (3) personality judgements of political candidates are drastically reduced to only two factors, despite many significant differences between their personalities.

The first of the two stable[8] personality factors for politicians has been named energy/innovation (which is a blend of energy and openness), and the second factor is honesty/trustworthiness (a blend of agreeableness, conscientiousness and stability).

These findings can be applied more generally, as shown by our replication study with 195 US college students. These students rated their own personalities after having rated Democratic President Bill Clinton and Republican presidential candidate Bob Dole, along with basketball star 'Magic' Johnson. The same 25 five-factor model marker adjectives were used as in our Italian study.

Table 1 shows that this different sample replicates the basic factor patterns found in the larger Italian sample: self-ratings and the ratings of the popular basketball player (among basketball fans) use all five factors, but judgements of the politicians are restricted to only three factors (among potential voters).

Finally, the percentage of total variance explained by each factor solution (2, 3 or 5) for each target personality, for both samples, is a high, nearly identical, average of 60 per cent.

We conclude that, by adopting a simplifying method of judging political candidates' personalities, voters use a cognitively efficient strategy for coding the mass of complex data, thus combating informational overload[9]. Doing so helps them to decide how to cast their vote.

Table 1 **Factor composition for target personalities**

	Factors					Variance explained (%)
	1	2	3	4	5	
Italian sample						
Self (*n*=827)	E	O	A	C	S	56
Athlete (*n*= 829)	A	E	C	O	S	57
TV star (*n*=830)	C	E	S	O	A	60
Politicians						
Berlusconi (*n*=1,257)	A+C+S	E+O	–	–	–	61
Prodi (*n*=643)	A+C+S	E+O	–	–	–	64
US sample						
Self (*n*=195)	A	C	E	O	S	57
Athlete (*n*=81)	S	C	O	A	E	61
Politicians						
Clinton (*n*=127)	E+O	C+A	?	–	–	57
Dole (*n*=127)	C+A	?	E+O	–	–	62

Factors 1-5 are arranged in order of the amount of variance in ratings, with Factor 1 explaining the most variance and Factor 5 the least. E, energy; O, openness; A, agreeableness; C, conscientiousness; S, emotional stability; and '?', an uninterpretable factor. Adjectives used to describe the politicians' two factors, normally attributed to the factors shown in parentheses, are – **energy/innovation:** enterprising (E), active (E), self-assured (E), energetic (E), cheerful (E), innovative (O), creative (O), inventive (O), smart (O), modern (O), efficient (C), optimistic (S), confident (S), cordial (A); and **honesty/trustworthiness:** sincere (A), truthful (A), loyal (A), responsible (C), reliable (C), precise (C), persistent (C), poised (S), peaceful (S), stable (S), generous (A).

Gian Vittorio Caprara*
Claudio Barbaranelli
University of Rome,
"La Sapienza",
Via dei Marsi 78,
Rome 1-00185, Italy
Philip G. Zimbardo
Psychology Department,
Stanford University,
Stanford,
California 94305, USA
e-mail: zim@psych.stanford.edu

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8. Tucker, L. R. *A Method for Synthesis of Factor Analysis Studies* (Dept of the Army, Washington DC, 1951).
9. Fiske, S. & Taylor, S. *Social Cognition* (McGraw Hill, New York, 1991).

*More detailed methods and additional results are available from G. V. C. at caprara@axrma.uniromal.it i)

§I Sets, spaces, and categories

1.1 Basics

sometimes denoted T^S

If S and T are sets, there is a set

$$F(S,T) = \{f : S \to T\}$$

of functions from S to T; a function

$$f : S \to T$$

is often identified with its graph

$$\mathrm{Gr}\, f = \{(s,t) \in S \times T \mid t = f(s)\}.$$

If $f : S \to T$ and $g : T \to U$ are functions, then $S \ni s \mapsto g(f(s)) := (g\circ f)(s) \in U$,

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & T \arrow[r, "g"] & U \end{tikzcd}$$

defines the composition of f with g. Equivalently:

$\exists$ map

$$F(S,T) \times F(T,U) \to F(S,U)$$
$$f, g \longmapsto g\circ f$$

of sets.

Proposition: The associativity diagram

$$\begin{tikzcd} F(S,T)\times F(T,U)\times F(U,V) \arrow[r] \arrow[d] & F(S,T)\times F(T,V) \arrow[d] \\ F(S,U)\times F(U,V) \arrow[r] & F(S,V) \end{tikzcd}$$

$$f,g,h \mapsto f, h\circ g$$
$$g\circ f, h \longmapsto h\circ g\circ f$$

is commutative.

Part ϕ : BACKGROUND

ii)

Def A function $f : S \to T$ of sets is an isomorphism if $\exists$ function $g : T \to S$ such that

1) $g\circ f = 1_S : S \to T \to S$ is the identity function ($1_S(s) = s$, $\forall s \in S$) of S, and

2) $f\circ g = 1_T : T \to S \to T$ is the identity function of T.

Notation

$$\begin{tikzcd} S \arrow[r, shift left, "f"] & T \arrow[l, shift left, "g"] \end{tikzcd}$$

$$S \cong T$$

Exercise: If such a g exists, it is unique.

Exercise The set

$$\mathrm{Aut}_{(\mathrm{sets})}(S,S) = \{f : S \to S \mid f \text{ is an isomorphism}\}$$

is a group.

Definition A set S is infinite if it is isomorphic with a proper subset of itself: ie if $\exists\, S_0 \subset S$, $S - S_0 \neq \emptyset$, together with an iso $S \to S_0$.

Ex $S = \mathbb{Z} = \{\ldots -2, -1, 0, +1, \ldots\}$ The integers

$S_0 = 2\mathbb{Z} = \{\ldots -4, -2, 0, +2, +4 \ldots\}$.

are isomorphic (via $S \ni x \mapsto 2x \in S_0$).

iii)

Ex If F is a finite set, its automorphism group Aut(F) is the symmetric group on #(F) elements

Definition, A set is finite if it is not infinite (Duh!

Proposition isomorphism is an equivalence relation: if $S \simeq T$ and $T \simeq U$, then $S \simeq U$. (Similarly: if $S \simeq T$ then $T \simeq S$, and $\forall\, S \simeq S$).

Proof If

$$\begin{tikzcd} S \arrow[r, shift left, "f"] & T \arrow[l, shift left, "f^{-1}"] \arrow[r, shift left, "g"] & U \arrow[l, shift left, "g^{-1}"] \end{tikzcd}$$

are isomorphisms, then

$$\begin{tikzcd} S \arrow[r, shift left, "g \circ f"] & U \arrow[l, shift left, "f^{-1} \circ g^{-1}"] \end{tikzcd}$$

is an isomorphism.

EQ relations & quotients

In particular, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$!

RECALL that if X is a set, and $\sim$ is an equivalence relation defined on X, then there is a function

$$q_\sim : X \to X/\sim \quad (= \text{a set of subsets of } X)$$

which sends $x \in X$ to $[x] = \{x' \in X \mid x \sim x'\}$ (its equivalence class)

Ex If A is an abelian group, and $B \subset A$ is a subgroup, $\exists$ eq. relation

$$a \sim a' \Leftrightarrow a - a' \in B$$

Defn

$$\begin{tikzcd} & X \times_Z Y \arrow[dl] \arrow[dr] & \\ X \arrow[dr, "f"] & & Y \arrow[dl, "g"'] \\ & Z & \end{tikzcd}$$

$X \times_Z Y = \{(x,y) \mid f(x) = g(y)\}$

= fiber product

$Z = pt$: $X \times Y$

f, g inclusions $\Rightarrow X \times_Z Y = X \cap Y$

iv)

(Check: $a \sim a \Leftrightarrow a - a = 0 \in B$
$a \sim b \Rightarrow b \sim a$? But B is a subgroup, so if $a - b \in B$, then $-(a-b) = b - a \in B$;
finally $a \sim b,\ b \sim c \Leftrightarrow a - b,\ b - c \in B$; but then

$(a-b) + (b-c) = a - c \in B \Rightarrow a \sim c$, QED).

Claim The map

$$q_\sim : A \to A/\sim := A/B$$

is a group homomorphism:

$$q_\sim(a_0 + a_1) = q_\sim(a_0) + q_\sim(a_1).$$

Ex. $A = \{(x_1, \dots, x_{n+m}) \mid x_i \in \mathbb{R}\} = \mathbb{R}^{n+m}$

$B = \{(0, \dots, x_{n+1}, x_{n+2}, \dots, x_{n+m}) \mid x_i \in \mathbb{R}\} \cong \mathbb{R}^m$,

$$\begin{tikzcd} B \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & A \arrow[r, "q_\sim"] \arrow[d, phantom, "\cong" rotate=90] & A/B \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{R}^m \arrow[r] & \mathbb{R}^{n+m} \arrow[r] & \mathbb{R}^n \end{tikzcd}$$

a surjective or onto group homomorphism

a one-to-one group homomorphism

Ex

$$0 \to B \to A \to A/B \to 0$$

is exact in the sense that each entry in the sequence (of groups and homomorphisms) the image of the incoming homomorphism equals the kernel of the outgoing homomorphism(s)

Ex

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X/A \arrow[dr, "p_B"'] & & X/B \arrow[dl, "p_A"] \\ & X/(A \cup B) & \end{tikzcd}$$

$Z \cong X/(A \cap B)$:

if $x \in X - A,\ y \in X - B$ then $p_B(x) = p_A(y) \Leftrightarrow x = y$;

if $x \in A$ or $y \in B$ then $p_B(x) = p_B(y) \Leftrightarrow x \in B \,\&\, y \in A$

Ex. $\forall n \in \mathbb{Z}$

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} := \mathbb{Z}_n \to 0$$

$$x \mapsto nx$$

is an exact sequence of groups & homomorphisms.

Definition Isomorphism defines an equivalence relation on the class of all sets. The class of equivalence classes of sets, under this relation, is the class of cardinal numbers. [Notation : $[S] := \#(S)$].

There is a set $\mathbb{N}$ of finite cardinal numbers, $= \{0, 1, \ldots\}$.

There is an abelian group

$$\mathbb{Z} := (\mathbb{N} \times \mathbb{N}) / \approx$$

$$\left[(a,b) \approx (a',b') \iff a + b' = a' + b, \text{ ie iff } a - b = a' - b'. \right]$$

1.2 Digression on abelian groups

homomorphism $\alpha : A \to B$ of abelian groups is a function such that $\alpha(a+a') = \alpha(a) + \alpha(a')$

Proposition If $\alpha : A \to B$, $\beta : B \to C$ are homomorphisms of abelian groups, then their composition $\beta \circ \alpha : B \to C$ is a homomorphism of abelian groups.

Proposition If $\alpha, \alpha' : A \to B$ are homomorphisms of abelian groups, then

$$(\alpha + \alpha')(a) := \alpha(a) + \alpha'(a)$$

defines a homomorphism $\alpha + \alpha' : A \to B$ of abelian groups.

Proof :

$$\begin{aligned} (\alpha + \alpha')(a+b) &= \alpha(a+b) + \alpha'(a+b) \\ &= (\alpha(a) + \alpha(b)) + (\alpha'(a) + \alpha'(b)) \\ &= (\alpha(a) + \alpha'(a)) + (\alpha(b) + \alpha'(b)) \\ &= (\alpha + \alpha')(a) + (\alpha + \alpha')(b) \end{aligned}$$

(uses both associativity & commutativity)

Def $\mathrm{Hom}(A,B) = \{$ homomorphisms from A to B $\}$ is an abelian group (with $0 : A \to B$ def'd by $0(a) = 0_B$).

vii)

Exercise If A, B, C are abelian groups, then the composition map

$$\alpha, \beta \longmapsto \beta\circ\alpha$$
$$\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \longrightarrow \mathrm{Hom}(A,C)$$

is a homomorphism of abelian groups

Moreover, if A, B, C, D are abelian groups, then the associativity diagram

$$\begin{tikzcd}
\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \times \mathrm{Hom}(C,D) \arrow[r] \arrow[dd] & \mathrm{Hom}(A,B) \times \mathrm{Hom}(B,D) \arrow[dd] \\
(\alpha, \beta, \gamma) \arrow[r, mapsto] \arrow[d] & (\alpha, \gamma\circ\beta) \\
\mathrm{Hom}(A,C) \times \mathrm{Hom}(C,D) \arrow[r] & \mathrm{Hom}(A,D) \\
(\beta\circ\alpha, \gamma) \arrow[r] & \gamma\circ\beta\circ\alpha
\end{tikzcd}$$

commutes:

Exercise Similarly, if V, W are vector spaces over a field F (i.e. abelian groups with a map

$$F \times V \longrightarrow V \qquad \text{such that} \qquad f_1, f_2 \in F,\ v \in V$$
$$f, v \longmapsto f\cdot v \qquad\qquad f_1\cdot(f_2 v) = (f_1 f_2) v$$
$$f(v_1+v_2) = fv_1 + fv_2$$

then the set

$$\mathrm{Hom}_F(V,W) = \{ \alpha \in \mathrm{Hom}(V,W) \mid \alpha(fv) = f\alpha(v) \},$$
$$V, W \in (F\text{-Vect})$$

is itself an F-vectorspace with $(f\cdot\alpha)(v) := f\alpha(v)$.

viii)

1.3 Topologies

RECALL that a topology $\mathcal{T}(X)$ on a set X is a collection $\mathcal{T} := \{U_\alpha \subset X\}$ of subsets of X, such that

1) $\emptyset, X \in \mathcal{T}$
2) if U_α, $\alpha \in I$ is a collection of open subsets of X then their union $\bigcup_{\alpha\in I} U_\alpha \subset X$ is an open subset of X
3) if U_α, $\alpha \in F$ is a finite collection of open subsets of X, then their intersection $\bigcap_{\alpha\in F} U_\alpha \subset X$ is an open subset of X.

A set X together with a topology $\mathcal{T}(X)$ is a topological space. (Ex. The collection of all subsets of X is a topology

Proposition: Suppose $\mathcal{T}_i$ is a collection of topologies on X. Then their intersection $\cap\mathcal{T}_i = \{ U \in \mathcal{T}_i \ (\forall i) \}$ is again a topology on X.

Proposition Suppose $\{\mathcal{O}_i \subset X\}$ is a family of subsets of X. There is a unique smallest topology $\mathcal{T}_{\mathcal{O}}$ on X, containing the sets $\mathcal{O}_i$, called the topology generated by $\{\mathcal{O}_i\}$.

ix)

Construction: Let $\{\mathcal{T}_\alpha\}$ be the collection of all topologies on X, such that each $\mathcal{T}_\alpha$ contains the sets $\{\theta_i\}$. This is a nonempty collection of topologies, and $\bigcap \mathcal{T}_\alpha$ is the smallest topology containing the open sets θ_i.

Ex. If $\underline{x} \in \mathbb{R}^n$, $\epsilon > 0$, then the open ϵ-balls

$$B^0_\epsilon(\underline{x}) = \{\underline{y} \in \mathbb{R}^n \mid |\underline{y} - \underline{x}| < \epsilon\}$$

around $\underline{x}$, $\forall \underline{x} \in \mathbb{R}^n$, generate the usual topology on X; similarly, for any metric space.

Ex: The product topology on $X \times Y$ is generated by $\{U \times V\}$, $U \subset X$ & $V \subset Y$ open

Definition A function $f : X \to Y$ between sets with topologies $\mathcal{T}(X), \mathcal{T}(Y)$ is continuous $\Leftrightarrow$ $\forall U \in \mathcal{T}(Y) \Rightarrow f^{-1}(U) = \{x \in X \mid f(x) \in U\} \in \mathcal{T}(X)$.

Exercise If $(X, \mathcal{T}(X)) \xrightarrow{f} (Y, \mathcal{T}(Y))$ and $(Y, \mathcal{T}(Y)) \xrightarrow{g} (Z, \mathcal{T}(Z))$ are continuous functions, then $g \circ f : (X, \mathcal{T}(X)) \to (Z, \mathcal{T}(Z))$ is continuous.

x)

Def If $(X, \mathcal{T}(X))$, $(Y, \mathcal{T}(Y))$ are topological spaces, ($\to$ sometimes denoted Y^X)

$$\mathrm{Maps}(X,Y) = \{\text{continuous fns from } (X, \mathcal{T}(X)) \text{ to } (Y, \mathcal{T}(Y))\}$$

[At this point I'll stop specifying the topology, and refer to these objects simply as topological spaces.]

Proposition If W, X, Y, Z are topological spaces, then the associativity diagram (of sets)

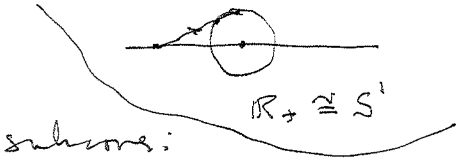
$$\begin{tikzcd}
\mathrm{Maps}(W,X) \times \mathrm{Maps}(X,Y) \times \mathrm{Maps}(Y,Z) \arrow[r] \arrow[d] & \mathrm{Maps}(W,X) \times \mathrm{Maps}(X \arrow[d] \\
\mathrm{Maps}(W,Y) \times \mathrm{Maps}(Y,Z) \arrow[r] & \mathrm{Maps}(W,Z)
\end{tikzcd}$$

commutes.

Note, spaces which are isomorphic in the sense of p. ii) are said to be homeomorp

Digression, on finiteness issues: As with sets, some spaces are bigger than others, and sometimes it is necessary to put finiteness restrictions on our spaces. Recall that a topological space is compact if it has the property, that every cover by

xi)



open sets has a finite subcover:

Ex. (Heine-Borel) a closed, bounded subset of $\mathbb{R}^n$ is compact

A space is locally compact if every point has a compact neighborhood; for example Euclidean space $\mathbb{R}^n$ is locally compact.

Note that any locally compact space X has a one-point compactification $X_+ = $ "$X \cup \infty$", with $\mathcal{T}(X_+)$ defined to be the topology generated by 1) the open sets of X, together with

2) sets of the form $(X - K) \cup \infty$, where K is compact in X.

The inclusion $X \hookrightarrow X_+$ maps X to the compact space X_+.

Exercise (Rotman Ch 0 p2: $\exists S^n \to \mathbb{R}^n_+$ homeomorphism

In this course we will usually deal only with Hausdorff spaces (in which distinct points $x, y \in X$ have neighborhoods $U_0 \ni x$, $U_1 \ni y$ such that $U_0 \cap U_1 = \phi$.

Def: Quotient topology $X \to X/\sim$

Hausdorff $\Leftrightarrow$ $R(\sim) \subset X \times X$ is closed

[Rotman].

xii)

Better solution: category of spaces htpy eq. to cell cxes! Rotman, Ch 8

Definition

Rotman Ch 11 (p 312-314)

Hatcher Appendix, p 529...

If $X \supset K$ compact and $Y \supset U$ open, let

$$\{K : U\} = \{f : X \to Y \mid f(K) \subset U\},$$

and let $\mathcal{T}(X,Y)$ be the topology on Maps(X,Y) generated by $\left\{ \{K:U\},\ \forall\, K \text{ compact in } X,\ U \text{ open in } Y \right\}$

Theorem If Y is locally compact, then the composition map

$$\text{Maps}(X,Y) \times \text{Maps}(Y,Z) \longrightarrow \text{Maps}(X,Z)$$

is continuous.

If, moreover, Z is Hausdorff, then

$$\text{Maps}(X \times Y, Z) \longrightarrow \text{Maps}(X, \text{Maps}(Y,Z))$$
$$\ni F \longmapsto (x \mapsto F(x \times -) : Y \to Z)$$

is a homeomorphism.

Corollary If X and Y are locally compact, then the associativity diagram in the proposition on p x) above is a commutative diagram of topological spaces and continuous maps.

xiii)

Refs: Kotman Ch 0 p 6–13
Hatcher Ch 2 § 3 p 162–165

(64) § Categories

A category is collection of objects (A, B, C, …), together with a set Maps(A,B) for each pair of objects, as well as composition functions

$$\mathrm{Maps}(A,B)\times\mathrm{Maps}(B,C)\to\mathrm{Maps}(A,C)$$

for each triple of objects, such that for each quadruple of objects, the associativity diagrams

$$\begin{tikzcd}
\mathrm{Maps}(A,B)\times\mathrm{Maps}(B,C)\times\mathrm{Maps}(C,D) \arrow[r] \arrow[d] & \mathrm{Maps}(A,B)\times\mathrm{Maps}(B,D) \arrow[d] \\
\mathrm{Maps}(A,C)\times\mathrm{Maps}(C,D) \arrow[r] & \mathrm{Maps}(A,D)
\end{tikzcd}$$

commute.

Moreover, for each object A there is an identity map $1_A \in \mathrm{Maps}(A,A)$, such that for any $f \in \mathrm{Maps}(A,B)$, $g \in \mathrm{Maps}(C,A)$, we have

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[rr, bend right, "f"'] & A \arrow[r, "f"] & B
\end{tikzcd}$$ commutes, ie $f = f\circ 1_A$

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[rr, bend right, "g"'] & A \arrow[r, "1_A"] & A
\end{tikzcd}$$ " , ie $g = 1_A\circ g$.

Ex cat of top. spaces & proper continuous maps $f: X\to Y$ ($K\subset Y$ compact $\Rightarrow f^{-1}(K)$ compact in X)
X, Y locally compact, f proper $\Rightarrow f_+ : X_+ \to Y_+$ is continuous.

xiii)

Examples: In the preceding we have defined the category (Sets) of Sets
(Ab) of abelian groups
(Top) of topological spaces.

These examples have some common properties: for example, the product operation

$$S, T \mapsto S\times T$$

on sets satisfies Cartan's identity

$$F(S\times T, U) \equiv F(S, F(T,U)).$$

Similarly the tensor product $A, B \mapsto A\otimes B$ of abelian groups satisfies

$$\mathrm{Hom}(A\otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B,C))$$

(and similarly, for the tensor product of the category of R-Modules, over a commutative ring R (eg the category of vector spaces over a field).

[In the category of topological spaces, on the other hand, something like this holds only under suitable finiteness conditions.]

Ex. fiber products of top spaces

Digression There are many variations on these themes. For example, a well-ordered set is a set S with an order relation such that

1) if $a \leq b$ and $b \leq c$ then $a \leq c$
2) $\forall a, b \in S$, either $a \leq b$ or $b \leq a$ (or, if both, then $a = b$),

with the further property that every subset of S has a least element with respect to this order.

There is a category of well-ordered sets, with monotone (or: order-preserving) functions as maps:

$$\mathrm{Maps}_{WO}(S,T) = \left\{ f \in F(S,T) \;\middle|\; \begin{array}{l} a \leq b \text{ in } S \\ \Rightarrow f(a) \leq f(b) \text{ in } T \end{array} \right\}$$

Isomorphism classes of objects in the category of well-ordered sets are called ordinal numbers.

Exercise Every category $\mathcal{C}$ has an opposite category $\mathcal{C}^{op}$, whose objects are the objects of $\mathcal{C}$, but with $\mathrm{Maps}_{\mathcal{C}^{op}}(A,B) = \mathrm{Maps}_{\mathcal{C}}(B,A)$.

Check that this does indeed define a category!

Historically, categories were invented to permit the definition of maps between them, called functors.

Algebraic Topology, roughly, studies topological spaces by associating algebraic objects (eg abelian groups) to them, in a natural way. This course is concerned largely with the (integral) homology functor

$$(\text{Spaces}) \ni X \longmapsto \bigoplus_{i \geq 0} H_i(X, \mathbb{Z}) := H_*(X, \mathbb{Z}) \in (\text{Ab})$$

and its variants.

Definition A functor $F : \mathcal{A} \to \mathcal{B}$ between categories

1) associates to every object A of $\mathcal{A}$, an object $F(A) \in \mathcal{B}$, and
2) associates to every $f : A \to$ in $\mathcal{A}$, a ie $f \in \mathrm{Map}_{\mathcal{A}}(A, A')$ an element $F(f) : F(A) \to F(A')$ in $\mathcal{B}$, ie $F(f) \in \mathrm{Map}_{\mathcal{B}}(F(A), F($

(alternately: $\exists$ function

$$\mathrm{Map}_{\mathcal{A}}(A, A') \xrightarrow{f \mapsto F(f)} \mathrm{Map}_{\mathcal{B}}(F(A), F(A')) \;)$$

such that

1) $F(1_A) = 1_{F(A)}$, ie F of the identity map of A is the identity map of F(A)

2) If $A \xrightarrow{f} A' \xrightarrow{g} A''$ in $\mathcal{A}$, then

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[rr, bend right, "F(g\circ f)"'] & F(A') \arrow[r, "F(g)"] & F(A'') \end{tikzcd}$$

commutes:

That is, $F(g\circ f) = F(g)\circ F(f)$ in $\mathcal{B}$.

Summed up as a slogan: functors preserve compositions of maps, in a "natural" way.

Examples An abelian group A is a set, with some extra structure (eg an identity element $0_A \in A$ and a composition law $A\times A \to A$). If we forget this extra structure, we get the "underlying set" of A. A homomorphism of abelian groups is also a map of sets, with extra properties, so forgetting those properties defines a functor

$$\begin{aligned} (Ab) &\longrightarrow (Sets) \\ A &\longmapsto \text{its underlying set} \end{aligned}$$

called the 'forgetful functor'.

Recall that the free abelian group generated by a set S

$$\mathbb{Z}[S] = \Big\{ \sum_{s\in S} a(s)\cdot s \Big\} \quad (= \oplus^S \mathbb{Z})$$

is the set of functions $a : S \to \mathbb{Z}$ which are 'almost always zero', ie $a(s) \neq 0$ only for finitely many s.

Ex If $S = \mathbb{Z}$ then $\mathbb{Z}[\mathbb{Z}]$ is isomorphic (as an abelian group) to the ring $\mathbb{Z}[x, x^{-1}]$ of Laurent polynomials: if $a : \mathbb{Z} \to \mathbb{Z}$ is almost always zero, then

$$\sum_{-\infty < n < +\infty} a(n)\, x^n \in \mathbb{Z}[x, x^{-1}].$$

Claim, if $a : S \to \mathbb{Z}$ is almost zero in the sense above, and $f \in F(S,T)$, then

$$\forall t \in T, \quad (f_* a)(t) = \sum_{f(s)=t} a(s) \in \mathbb{Z}$$

defines an almost-zero function $f_* a : T \to \mathbb{Z}$

Proof; There is a finite set

$$S_{\neq 0} := \{ s \in S \mid a(s) \neq 0 \} \subset S$$

such that $a = 0$ on $S - S_{\neq 0}$. Now $f : S \to T$ defines a partition

$$S = \bigcup_{t \in T} f^{-1}(t)$$

of S into disjoint subsets (since f is a function). Now if $(f_* a)(t) \neq 0$ $\exists\, s \in S_{\neq 0}$ such that $f(s) = t$, ie $S_0 \cap f^{-1}(t)$ is nonempty; but there can be only finitely many such t, because the $f^{-1}(t)$ are disjoint.

Corollary $S \mapsto \mathbb{Z}[S] : (\text{Sets}) \longrightarrow (\text{Ab})$ is a functor.

Remark $S \mapsto F(S, \mathbb{Z}) = \mathbb{Z}^S \; (= \prod^S \mathbb{Z})$ defines a functor $(\text{Sets}) \longrightarrow (\text{Ab})^{op}$: if $f \in F(S,T)$ then

$$f^* : \mathbb{Z}^T = F(T, \mathbb{Z}) \longrightarrow F(S, \mathbb{Z}) = \mathbb{Z}^S,$$

(ie $f^*[T \xrightarrow{b} \mathbb{Z}] := [S \xrightarrow{b} T \to \mathbb{Z}]$) is such that if $g \in F(T, U)$, then

$$(g \circ f)^* = f^* \circ g^* .$$

Example $V \longmapsto \text{Hom}_F(V, F) \overset{:= V^*}{}$ is a functor from (F - Vector Spaces) to (F - Vector Spaces)^{op}.

It takes the subcategory of finite-dimensional vector spaces to its opposite category.

Exercise This restriction is an equivalence of $(F\text{-Vect})_{\text{finite-dim'l}}$ with its opposite category.

[If we think of linear transformations as matrices, this just sends a matrix to its transpose.]

(when V is finite-dim'l!)

Hint: $\exists$ isomorphism $V \longrightarrow (V^*)^*$ defined by $V \ni v \longmapsto [V^* \ni w \longmapsto w(v) \in F] \in (V^*)^*$.

✧ Example $\boxed{\text{Rotman [p 27, Ch 1.16]}}$ defines a functor

$$\pi_0 : (\text{Spaces}) \longrightarrow (\text{Sets})$$

which sends a space X to its set $\pi_0(X)$ of path-components.

Def'n A Riemannian metric g on a (f.dim'l) vector space V is a choice $g : V \to V^*$ of a vector space isomorphism, such that $g \in \text{Hom}(V, \text{Hom}(V, \mathbb{R})) \cong \text{Hom}(V \otimes V, \mathbb{R})$ satisfies $g(v,w) = g(w,v)$. It follows that $g(v,v) \geq 0$ and that $g(v,v) = 0 \Rightarrow v = 0$, ie V is an inner product space.

Remark. A space X is connected if it is not the disjoint union of two (nontrivial) open subsets. This defines an equivalence relation on X : $x \sim y$ in X if $\exists$ connected $C \subset X$, such that $x, y \in C$. The equivalence classes of this relation are the components of X. The set of components is, in general, a quotient of the set of path components.

Claim The zeroth homology group

$$H_0(X, \mathbb{Z}) = \mathbb{Z}[\pi_0(X)]$$

of a space is the free abelian group generated by its set of path components.

[This is a consequence of Th 4.13 (p 68) of Rotman].

Definition Let $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ be the class of functors from $\mathcal{A}$ to $\mathcal{B}$: then there are composition maps

$$\mathrm{Fun}(\mathcal{A}, \mathcal{B}) \times \mathrm{Fun}(\mathcal{B}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{C})$$

These are not sets in general, so the class of categories is not itself a category

If $F, G : \mathcal{A} \to \mathcal{B}$ are functors, there is a notion of a natural transformation $\tau : F \to G$, i.e. $\forall A \in \mathcal{A}\ \exists\ \tau(A) : F(A) \to G(A)$, such that $\forall f \in \mathrm{Map}_{\mathcal{A}}(A, A')$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\tau(A)"] & F(A') \arrow[d, "\tau(A')"] \\ G(A) \arrow[r, "G(f)"] & G(A') \end{tikzcd}$$

commutes.

The class $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ thus becomes i.e. it is the class of objects a category, with natural transformations of functors as maps. [Rotman Ch 9 p 228]. The category of categories is thus enriched over itself in a certain sense.

Ex If $\mathrm{Id}(V) = V$, and $\mathrm{Id}^{**}(V) = (V^*)^*$, then (p xx) $\exists$ natural transformation $\mathrm{Id} \to \mathrm{Id}^{**}$ in $\mathrm{Fun}(\mathrm{Vect}, \mathrm{Vect})$.

§1.5 A little more algebra

A *directed set* $(I, \leq)$ is a set with a transitive, reflexive $(a \leq a)$ relation with upper bounds: $\forall a,b \in I\ \exists\ c \geq a,b$. A directed set can be regarded as a *category* $\underline{\underline{I}}$, with

$$\mathrm{Maps}_I(a,b) = \emptyset \quad \text{if } a \text{ is not} \leq b$$
$$= \text{a singleton otherwise};$$

and a functor $\underline{\underline{\Phi}} : \underline{\underline{I}} \to \mathbb{C}$ is called a directed (sometimes inductive) system.

Ex The set $\mathfrak{T}_x(X) = \{U \text{ open} \subset X \mid x \in U\}$ of open subsets of a space X, all containing a point x, is directed, if $U \leq V$ means $V \supset U$.

Ex The positive integers $\mathbb{N}_\times$ is a directed set, if $n \leq m$ means $m = nk$ for some $k \in \mathbb{N}$, ie n *divides* m.

In the first example, $U \to \mathrm{Maps}(U, \mathbb{R}) := C(U)$ is a directed family of commutative $\mathbb{R}$-algebras. In the second,

$$n \mapsto q(n) = \mathbb{Z},$$
$$n \leq m \mapsto \text{mult by } k = m/n:$$
$$q(n) = \mathbb{Z} \to q(m) = \mathbb{Z}$$

is a directed family of abelian groups.

Definition, In a category $\mathbb{C}$ with arbitrary sums, the directed limit

$$\varinjlim_I \Phi = \coprod_{i \in I} \Phi(i) \,/\, (\text{relations})$$

(nowadays more usually called the (directed) colimit) defined by relations

$a \in \Phi(i) \ni a \sim a' \in \Phi(i') \iff$

$\exists k \geq i, i'$ and $\alpha \in \Phi(k)$ such that

$\alpha \mapsto a$ in $\Phi(i)$, $\alpha \mapsto a' \in \Phi(i')$.

In other words, $\varinjlim_{I} \Phi$ fits in a diagram

$$\begin{tikzcd}
{} \arrow[d] \arrow[dr] & \\
\Phi(i) \arrow[r] \arrow[d, "i \geq i'"] & \varinjlim_{I} \Phi \\
\Phi(i') \arrow[ur] \arrow[d] & \\
{} &
\end{tikzcd}$$

such that any family of maps from the system Φ to an object of $\mathcal{C}$ factors (<u>uniquely</u>) through $\varinjlim_{I} \Phi$.

<u>Ex</u>. $\varinjlim_{\mathcal{N}_x(X)} C(U) =$ the ring $\hat{\mathcal{C}}_x(X)$ of 'germs' of functions defined on a neighborhood of x

<u>Ex</u> $\varinjlim_{\mathbb{N}} 9 = \mathbb{Q}$ is the ring of rational numbers,

$$\begin{tikzcd}
{} \arrow[d] \arrow[r] & {} \\
\mathbb{Z} \arrow[d, "n^{-1}m"] \arrow[dr, "m^{-1}"] & \\
\mathbb{Z} \arrow[r, "n^{-1}"] \arrow[d] & \mathbb{Q} \\
\mathbb{Z} &
\end{tikzcd}$$

<u>Definition</u>, A sequence

$$\Phi' \to \Phi \to \Phi''$$

of directed systems of abelian groups (or R-modules) is <u>exact</u> if $\forall i \in I$, the sequence

$$\Phi'(i) \to \Phi(i) \to \Phi''(i)$$

of abelian groups is exact at $\Phi(i)$.

<u>Claim</u>, If $\Phi' \to \Phi \to \Phi''$ is exact in this sense then

$$\varinjlim_{I} \Phi' \to \varinjlim_{I} \Phi \to \varinjlim_{I} \Phi''$$

is exact.

<u>Remark</u>, In other words the functor

$\varinjlim_{I}$: (<u>Directed systems</u> of $\mathbb{Z}$-Mod (or R)) $\to$ <u>R-Mod</u>

is *exact*. In general it is *not* the case that on R-Modules various (interesting, natural) functors – eg $\mathrm{Hom}(-,B)$, $\mathrm{Hom}(B,-)$, $-\otimes B$ ... are exact. How to deal with this is the concern of *homological algebra* – which I will avoid as much as possible in these notes, leaving it to be discussed systematically in *Alg Top II*.

Remark There is a dual notion of *projective* or *inverse* limit of a directed system,

$$\varprojlim_{I} \Psi = \{\alpha \in \prod \Psi(i) \mid \text{if } i' \geq i \text{ then } \alpha(i') \to \alpha(i)\}.$$

Example $\mathbb{Z}/p \xleftarrow{(\text{mod } p)} \mathbb{Z}/p^2 \leftarrow \cdots \leftarrow \mathbb{Z}/p^n\mathbb{Z} \leftarrow \cdots$

is an inverse system, with the ring

$$\mathbb{Z}_p := \{\sum a_i p^i, \ a_i \in \mathbb{Z}/p\}$$

of

p-adic integers as its projective limit. *Note*, that as a closed subset of the product $\prod_{n\geq 1} \mathbb{Z}/p^n$ of finite sets, $\mathbb{Z}_p$ is compact in a natural topology.

Inverse limits in general do *NOT* preserve exactness. Rather than pursue that here, I'll just note that *Pontryagin duality* asserts that the category of locally compact abelian groups is *equivalent* to its opposite category, via the $\boxed{\text{contravariant}}$ functor sending A to the (locally compact topological) *character* group

$$A^\vee = \mathrm{Hom}_c(A, \mathbb{T})$$

of continuous homomorphisms to the circle group $\mathbb{T} = \{u \in \mathbb{C} \mid |u| = 1\}$.

Under this (exactness-preserving) equivalence the opposite of the category of *discrete* abelian groups is the category of *compact* topological abelian groups.

The Pontrjagin dual of an inductive system of abelian groups is a projective system of compact groups, so the inverse limit of such a system *does* preserve exactness. For example the projective limit of the system

$$z \mapsto z^n : \mathbb{T} \to \mathbb{T}$$

of coverings of the circle, ordered by divisibility, is the *solenoidal* group $\Sigma = \varprojlim \mathbb{T}$, dual to the discrete abelian group $\mathbb{Q}$.

We will also encounter the fact that the pairing

$$(\oplus_S \mathbb{Z}) \times (\prod^S \mathbb{Z}) \longmapsto \mathbb{Z}$$

which sends a sequence $a_i \in \mathbb{Z}$, $a_i = 0$ for $i \gg 0$ and a sequence $b_i \in \mathbb{Z}$, *unrestricted*, to

$\sum a_i b_i \in \mathbb{Z}$, is perfect: that is, that

$$\prod^S \mathbb{Z} \to \mathrm{Hom}(\oplus_S \mathbb{Z}, \mathbb{Z})$$

is an isomorphism (and, dually).

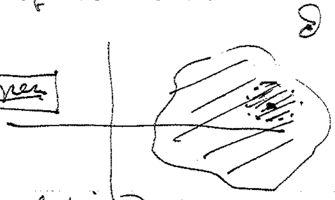
§II Background from Geometry

i)

§ Tangent Spaces

Let $\mathcal{O} \subset \mathbb{R}^n$ be an open subset of Euclidean space, $\underline{x} \in \mathcal{O}$ a point in it.

mathcal $\mathcal{O}$ for open



It makes sense to say that f is a real-valued function defined in some open neighborhood of $\underline{x}$, and therefore that f is differentiable (C^1) or even smooth (C^∞) at $\underline{x}$.

Definition: The tangent space $T_{\underline{x}}\mathcal{O}$ of $\mathcal{O}$ at $\underline{x}$ is the real vector space of linear operators

$$(a, b \in \mathbb{R}) \quad L(af_1 + bf_2) = aL(f_1) + bL(f_2) \in \mathbb{R}$$

from such functions to the reals, which satisfy Leibniz's rule

$$L(f_1 \cdot f_2) = f_1(\underline{x}) \cdot Lf_2 + f_2(\underline{x}) \cdot Lf_1 \in \mathbb{R}.$$

Proposition If $L \in T_{\underline{x}}\mathcal{O}$ then

$$L(f) = \sum_{i=1}^{i=n} v_i(L) \frac{\partial f}{\partial x_i}(\underline{x})$$

for some unique $\underline{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$. In other words, $T_{\underline{x}}\mathcal{O} \cong \mathbb{R}^n$ as linear vector spaces

ii)

This extends to an isomorphism

$$(L, \underline{x}) \mapsto (\underline{v}(L), \underline{x}) :$$

$$T(\mathcal{O}) := \coprod_{\underline{x} \in \mathcal{O}} T_{\underline{x}}\mathcal{O} \xrightarrow{\cong} \mathbb{R}^n \times \mathcal{O}$$

(of parametrized vector spaces).

Suppose further that

$$\mathbb{R}^n \supset \mathcal{O} \ni \underline{x} = (x_1, \ldots, x_n) \xrightarrow{\underline{F}} (F_1(\underline{x}), \ldots, F_m(\underline{x})) \in \mathcal{O}' \subset \mathbb{R}^m$$

is a smooth (C^∞) function, between open sets in Euclidean spaces (not necessarily of the same dimension). Then the Jacobian matrix

$$\underline{F}'(\underline{x}) = \left[\frac{\partial F_i}{\partial x_k}(\underline{x}) \right]$$

defines a map

$$\begin{tikzcd} \mathbb{R}^n \times \mathcal{O} \ni (\underline{v}, \underline{x}) \arrow[r, mapsto] \arrow[d] & (\underline{F}'(\underline{x}) \cdot \underline{v}, \underline{F}(x)) \in \mathbb{R}^m \times \mathcal{O}' \arrow[d] \\ T(\mathcal{O}) \arrow[r, "T(F)"] & T(\mathcal{O}') \end{tikzcd}$$

(of families of parametrized vector spaces).

This is just a form of the chain rule:

$$\underline{v} \mapsto \sum_{k=1}^{k=n} v_k \frac{\partial F_i}{\partial x_k}(\underline{x}) \quad (i = 1, \ldots, m).$$

iii)

Milnor, Topology from a differentiable viewpoint
Spivak, Calculus on manifolds

More generally, if

$$\mathcal{O} \xrightarrow{F} \mathcal{O}' \xrightarrow{G} \mathcal{O}''$$

are smooth maps of open subsets of Euclidean space, the *chain rule* becomes the assertion that the diagram

$$\begin{tikzcd}
T(\mathcal{O}) \arrow[r, "T(F)"] \arrow[d, "\cong"] & T(\mathcal{O}') \arrow[r] \arrow[d, "\cong"] & T(\mathcal{O}'') \arrow[d, "\cong"] \\
\mathbb{R}^n \times \mathcal{O} \arrow[r, dashed] & \mathbb{R}^m \times \mathcal{O} \arrow[r, dashed] & \mathbb{R}^p \times \mathcal{O}'
\end{tikzcd}$$

commutes: since $(G \circ F)'(\underline{x}) = F'(G(\underline{x})) \cdot G'(\underline{x})$,

matrix product!

we have $T(G \circ F) = T(G) \circ T(F)$

Defn The family $T(\mathcal{O}) \to \mathcal{O}$ (of vector spaces parametrized by $\mathcal{O}$), ie the *tangent bundle* of $\mathcal{O}$, defines a *functor* from the category of open subsets of Euclidean space, and smooth maps between them, to the category of *vector bundles*: *roughly*, the category of parametrized families of vector spaces, of locally constant dimension — with parametrized families of linear transformations as maps.

iv)

(22) § The implicit fn. theorem

The *inverse* function theorem says that if

$$\mathbb{R}^n \supset \mathcal{O} \xrightarrow{F} \mathcal{O}' \subset \mathbb{R}^n$$

is a *smooth* map between Euclidean opens *of the same dimension*, and if $\underline{x}_0 \in \mathcal{O}$ is a point such that

$$T_{\underline{x}_0} F : T_{\underline{x}_0} \mathcal{O} \xrightarrow{\cong} T_{F(\underline{x}_0)} \mathcal{O}'$$

is a vector space *isomorphism* (ie, if the determinant of the Jacobian matrix of F at $\underline{x}_0$ is nonvanishing), then there are open neighborhoods V of $\underline{x}_0$ and V' of $\underline{y} = F(\underline{x}_0)$, together with a *smooth inverse function* $F^{-1} : V' \xrightarrow{\cong} V$, ie such that

$$F^{-1}(F(\underline{x})) = \underline{x} \quad \text{for } \underline{x} \in V$$
$$F(F^{-1}(\underline{y})) = \underline{y} \quad \text{for } \underline{y} \in V'.$$

Marsden & Tromba
Vector Calculus
§3.5 p226-2

Examples: exp & log, tan & arctan . . .

The general *implicit function theorem* can be deduced quickly from the theorem above, which is a special case

Its proof is very similar to that of the following corollary, which requires a definition to state.

Definition $y_0 \in \mathcal{O}' \subset \mathbb{R}^m$ is a regular value of a smooth function

$$\mathbb{R}^{n+m} \supset \mathcal{O} \xrightarrow{F} \mathcal{O}' \subset \mathbb{R}^m$$

if $\exists\ \underline{x}_0$ such that $F(\underline{x}_0) = \underline{y}_0$, and such that

$$T_{\underline{x}_0} F : T_{\underline{x}_0} \mathcal{O} \longrightarrow T_{\underline{y}_0} \mathcal{O}'$$

is a surjective (= onto) linear map.

Theorem If y_0 is a regular value of F, with $\underline{x}_0 = (\underline{u}_0, \underline{v}_0) \in \mathbb{R}^n \times \mathbb{R}^m$ as above, then $\exists$ a smooth function

$$\mathbb{R}^n \supset \mathcal{O}_1 \xrightarrow{g} \mathcal{O}_2 \subset \mathbb{R}^m, \quad \underline{u}_0 \in \mathcal{O}_1, \ \underline{v}_0 \in \mathcal{O}_2, \ \mathcal{O}_1 \times \mathcal{O}_2 \subset \mathcal{O}$$

such that $g(\underline{u}_0) = \underline{v}_0$, and such that the graph

$$\mathcal{O}_1 \ni \underline{u} \mapsto (\underline{u}, g(\underline{u})) \in \mathcal{O}_1 \times \mathcal{O}_2 \subset \mathcal{O}$$

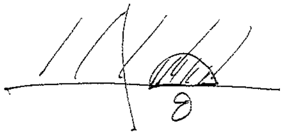
maps an open neighborhood of $\underline{u}_0$ isomorphically to an open neighborhood of $\underline{x}_0 = (\underline{u}_0, \underline{v}_0) \in F^{-1}(\underline{y}_0)$.

In particular, $F(\underline{u}, g(\underline{u})) = \underline{y}_0$, for $\underline{u}$ near $\underline{u}_0$

Ex. $F(x,y) = x^2 + y^2 : \mathbb{R}^2 \to \mathbb{R}$ has $1 \in \mathbb{R}$ as regular value, and

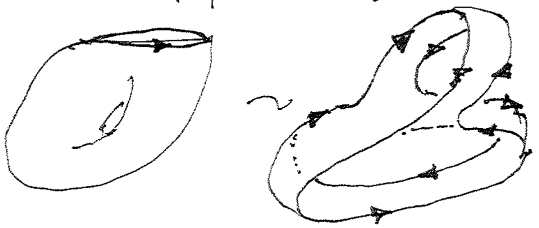
$$g(x) = \sqrt{1 - x^2} : \mathbb{R} \to \mathbb{R} \qquad \text{for } \underline{x}_0 = (0,1).$$

Remark A (deep & useful) Theorem of Sard (Milnor, Topology from a diff'bl viewpoint, §2 p 8) says that in the situation above, almost every $\underline{y} \in \mathbb{R}^m$ is a regular value.



2.3 **Manifolds** M

A manifold is a (at least) locally Euclidean space: $\forall\, x \in M$ $\exists$ open neighborhood $U \ni x$ together with a homeomorphism $\varphi : U \to \mathcal{O} \subset \mathbb{R}^n$ with an Euclidean open set. [More generally: with an open set in some Euclidean half-space $\{\underline{x} \in \mathbb{R}^n \mid x_n \geq 0\}$.

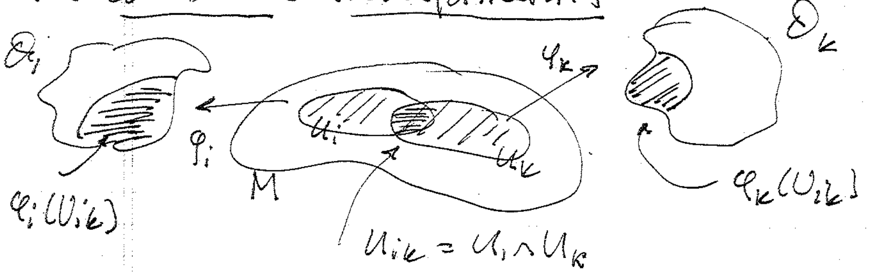


More formally: a smooth manifold is a topological space M with an Euclidean atlas : an open cover

$$\{ M \supset U_i \xrightarrow{\varphi_i} \Theta_i \subset \mathbb{R}^n \} \quad , \quad M = \cup U_i ,$$

$\varphi_i \in$ homeomorphism,

coordinate patches, such that the coordinate transformations



$$\varphi_i \circ \varphi_k^{-1} := \Phi_{ik} : \mathbb{R}^n \supset \varphi_k(U_{ik}) \longrightarrow \varphi_i(U_{ik}) \subset \mathbb{R}^n$$

are C^∞ ! , and such that, for $x \in U_{ijk} = U_i \cap U_j \cap U_k$,

we have the cocycle condition

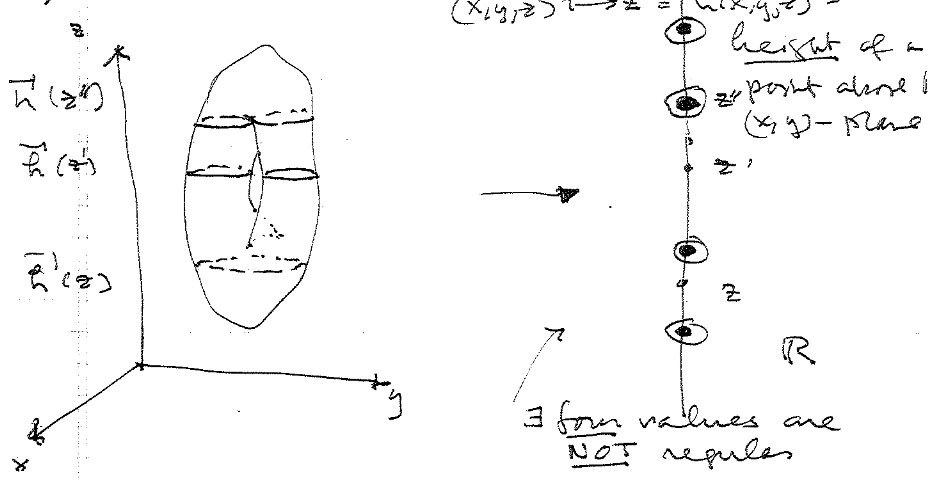
$$\Phi_{ik} \circ \Phi_{kj} \circ \Phi_{ji} = \text{identity} .$$

A map $M \longrightarrow M'$ of topological spaces (with manifold structure) is a map of manifolds (or, more simply, smooth) iff the compositions

$$\Theta_i \supset \varphi_i^{-1}(U_i \cap F^{-1}(U'_k)) \xrightarrow{F \circ \varphi_i} F(U_i) \cap U'_k \xrightarrow{\varphi'_k} \Theta'_k$$

are smooth ($\forall i,k$).

A 2-dim torus in M



We can define the tangent space $T_x M$ to a smooth manifold at $x \in M$ as above, in terms of differentiation operators on smooth functions defined in an open neighborhood of x.

It follows from the arguments above that there is a commutative diagram

$$\begin{tikzcd}
T(U_i) \arrow[r, "T(\varphi_i)"] \arrow[d, "\cong"] & T(\Theta_i) \arrow[d, "\cong"] \\
\mathbb{R}^n \times U_i \arrow[r] & \mathbb{R}^n \times \Theta_i
\end{tikzcd}$$

, and that

if you look closely, this is a tautology: it defines $T(U_i)$ to have the right proper

$$T(\Phi_{ik}) \cdot T(\Phi_{kj}) \cdot T(\Phi_{ji}) = n \times n \text{ identity matrix} .$$

§2.4 Submanifolds & Transversality

A smooth map $M \xrightarrow{F} M'$ of manifolds is an <u>immersion</u> if $\forall m \in M$, the tangent map

$$T_xF : T_xM \longrightarrow T_{F(x)}M'$$

is one-to-one (ie has kernel zero). F is an <u>embedding</u> if is one-to-one as a map of <u>sets</u>

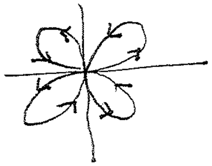
<u>Ex</u> The 'one-dimensional torus'

$\mathbb{Z} \overset{\cong}{\longmapsto} \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$

$t \mapsto \exp(2\pi i t)$ is a manifold,

and the 2k-petalled if $k \in \mathbb{Z}$ is <u>odd</u>
k-petalled if $k \in \mathbb{Z}$ is <u>even</u> } <u>Rose</u>

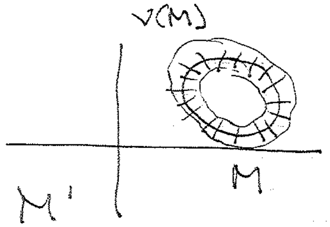
$$\begin{aligned} \mathbb{T} &\longrightarrow \mathbb{C} \\ t &\longmapsto \exp(2\pi i t)\cdot \cos 2\pi k t \end{aligned}$$



is an immersion of the circle in the plane, which is <u>not</u> an imbedding.

If $M \to M'$ is a smooth embedding (say of an n-manifold in an $n' > n$-dimensional manifold) then the <u>normal</u> <u>bundle</u>

$$\begin{tikzcd} \coprod_{x \in M} \operatorname{coker}[T_xf : T_xM \to T_{f(x)}M'] := \nu(f) \arrow[d] \\ M \end{tikzcd}$$



of the embedding f is family of $(n'-n)$-dim'l M' vector spaces parametrized by M.

<u>Definition</u> The <u>fiber</u> <u>product</u> (of $f : X \to Z$, $g : Y \to Z$) is the topological space

$$\begin{tikzcd} X \times_Z Y \arrow[r, dashed] \arrow[d, dashed] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

$$X \times_Z Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\}.$$

<u>Ex</u>. If $Z = pt$ then $\forall f, g$ $X \times_Z Y = X \times Y$

if f, g are <u>inclusions</u> then $X \times_Z Y \cong X \cap Y \subset$

<u>Useful</u> <u>fact</u> if $V \xrightarrow{\pi} Y$ is an n-dimensional vector bundle over Y, and $f : X \to Y$ is a continuous map, then the fiber product

$$\begin{tikzcd} f^*V = X\times_Y V \arrow[r] \arrow[d] & V \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is again an n-dimensional vector bundle, now on X.

Claim Any embedding $f: M \to M'$ defines an exact sequence

$$0 \to T(M) \longrightarrow f^*T(M') \longrightarrow \nu(f) \to 0$$

of vector bundles over M. That is, at each $x \in M$ the sequence

$$0 \longrightarrow T_x M \xrightarrow{T(f)} T_{f(x)} M' \longrightarrow \nu_x(f) = \operatorname{coker} T_x(f) \longrightarrow 0$$

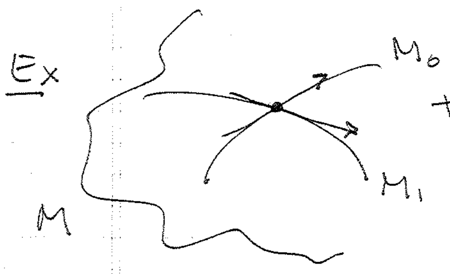
is exact (more or less by definition).

Definition If $M_0, M_1 \xrightarrow{f_0, f_1} M$ are embeddings of manifolds, and $x \in f_0(M_0) \cap f_1(M_1)$ is a point in the intersection of their images, then M_0 and M_1 intersect transversally at x iff the linear transformation

$$T_{x_0}M_0 \oplus T_{x_1}M_1 \longrightarrow T_x M \qquad (f(x_0) = f(x_1) = x)$$
$$(v_0, v_1) \longmapsto T_{x_0}f_0(v_0) + T_{x_1}f_1(v_1)$$

is surjective (onto).

Ex



the tangent vectors of the intersecting curves span the tangent space of the ambient manifold.

More generally: two maps $f_0 : M_0 \to M_\alpha$, $f_1 : M_1 \to M$ (not necessarily embeddings) are transversal at $x \in f_0(M_0) \cap f_1(M_1)$ if the corresponding map $T_{x_0}M_0 \oplus T_{x_1}M_1 \longrightarrow T_x M$ is similarly surjective.

Thom's transversality theorem is a very general version of the implicit function theorem:

i) $\forall$ smooth $f_0 : M_0 \to M_\ast$, $f_1 : M_1 \to M$

$\exists$ diagram

$$f^*T(M_0) \oplus f^*T(M_1) \xrightarrow{T(f_0) \oplus T(f_1)} h^*T(M)$$

on the pullback

$$\begin{tikzcd} & M_0 \times_M M_1 \arrow[dl, "\bar{f}_1"'] \arrow[dr, "\bar{f}_0"] \arrow[dd, dashed, "h"] & \\ M_0 \arrow[dr, "f"'] & & M_1 \arrow[dl, "f_1"] \\ & M & \end{tikzcd}$$

. If f_0 and f_1 are transversal, in the sense that this vectorbundle homomorphism is onto, then the fiber product $M_0 \times_M M_1$

is a *manifold*, and its tangent bundle is the *kernel* of the homomorphism $T(f_0)\oplus T(f_1)$ of vector bundles.

Moreover (under reasonable conditions of compactness and dimension), f_0 and f_1 are *almost always* transversal: that is, for any pair (f_0,f_1) of maps as above, there is a *transversal* pair $(\tilde{f}_0,\tilde{f}_1)$ arbitrarily close!

Ex. If V_0, V_1 are linear subspaces of a vector space W (of dimensions v_0, v_1, w respectively) then their intersection $V_0\cap V_1 \subset W$ is generically of dimension v_0+v_1-w : eg $v_0=v_1=2$, $w=3$ $\Rightarrow$ planes in $\mathbb{R}^3$ intersect generically in a *line*

Exercises: 1) $(X\times_Y Z)\times_Z W \simeq X\times_Y W$ (if the maps are the right ones!)

2) A *correspondence* $A : X \dashrightarrow Y$ is a map $A \to X\times Y$. Show that correspondences $A : X \dashrightarrow Y$, $B : Y \dashrightarrow Z$ compose associatively, via the diagram

$$\begin{tikzcd}
 & & A\times_Y B \arrow[dll] \arrow[d] \arrow[drr, bend left] & & \\
A \arrow[d] & & X\times Y\times Z \arrow[dll] \arrow[d] \arrow[drr] & & B \arrow[d] \\
X\times Y \arrow[drr, bend right] & & X\times Z & & Y\times Z \arrow[dll] \\
 & & Y & &
\end{tikzcd}$$

㉕ *Examples of manifolds*

i) Manifolds, as defined above, are locally compact, but further finiteness conditions are usually required. For example the *long line* is the one-dimensional manifold defined by the order topology on the set $\omega \times \mathbb{R}$ given the dictionary order $[\,(a,b) \le (c,d)$ if $a \le c$, or ~~a~~ if $a = c$ and $b \le d\,]$: where ω is the first uncountable ordinal. The long line is a manifold with boundary, but it is not metrizeable.

2) If M is a manifold, and $C \subset M$ is a closed subset, then $M - C$ is a manifold.

3) The space

$$\{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1 \} \cong S^3$$

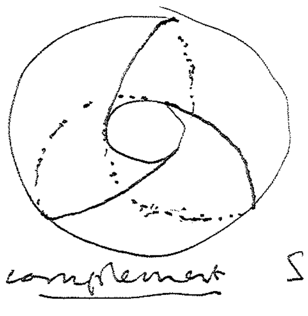
is isomorphic to the three-sphere, regarded as a subset of $\mathbb{R}^4$, and the 2-torus

$$T^2 = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = 1/\sqrt{2} \, , \, |z_2| = 1/\sqrt{2} \}$$

is a submanifold. If $(p,q) = 1$ are coprime integers, then

$$\mathbb{R}/\mathbb{Z} \ni t \mapsto 2^{-1/2} (\exp(2\pi i p t), \exp(2\pi i q t)) \in T^2$$

embeds the circle in $S^3 \cong \mathbb{R}^3_+$ as a (p,q) torus knot



In particular the knot complement $S^3 - k$ is a manifold.

Ex 4) The thick diagonal

$$\mathbb{C}^n \supset \Delta_n = \{ \underline{z} \in \mathbb{C}^n \mid \exists\, i \neq k \text{ such that } z_i = z_k \}$$

is a finite union of hyperplanes and is hence closed, so the configuration space

$$\mathrm{Config}^n(\mathbb{C}) = \mathbb{C}^n - \Delta_n = \{ \underline{z} \in \mathbb{C}^n \mid i \neq k \Rightarrow z_i \neq z_k \}$$

of ordered distinct n-tuples of points in the plane is a manifold.

(36)

Group actions

Rotman Ch 10
p280, 290-292
see also Ch 8 p 180,
Ch 1 p 18-22
on quotient spaces

An action

$$a : G \times X \to X$$

of a group G on a set X, usually written $a(g,x) := g \cdot x$ is a map such that $(g_0 \overset{\text{group mult!}}{\cdot} g_1) \overset{\text{action}}{\cdot} x = g_0 \overset{\text{action}}{\cdot} (g_1 \cdot x)$; alternately, such that the associativity diagram

$$\begin{tikzcd} G \times G \times X \arrow[r, "1_G \times a"] \arrow[d, "\mu \times 1_G"'] & G \times X \arrow[d, "a"] \\ G \times X \arrow[r, "a"'] & X \end{tikzcd}$$

commutes: where $\mu : G \times G \to G$ is the group multiplication.

Ex. If $\Sigma(n) = \mathrm{Aut}$ (set with n elements) is the symmetric group on n letters (Recall $\# \Sigma(n) = n!$), then

$$\Sigma(n) \times \mathbb{C}^n \to \mathbb{C}^n$$
$$\sigma, \underline{z} = (z_1, \dots, z_n) \mapsto (z_{\sigma(1)}, \dots, z_{\sigma(n)})$$

is a group action. It is a continuous, linear action: each $\sigma \in \Sigma(n)$ defines a continuous, linear map $\mathbb{C}^n \to \mathbb{C}^n$.

In general we will consider continuous actions, ie such that the map a is continuous, eg with the discrete topology on G.

Ex $\Sigma(n) \times \mathrm{Config}^n(\mathbb{C}) \to \mathrm{Config}^n(\mathbb{C})$

Ex. If $\mathbb{C}_0^{n-1} = \{ \underline{z} \in \mathbb{C}^n \mid \Sigma z_i = 0 \} \subset \mathbb{C}^n$ then the action of $\Sigma(n)$ on $\mathbb{C}^n$ restricts to an action on $\mathbb{C}_0^{n-1}$ (called the *reduced regular representation* of $\Sigma(n)$).

Definition If G acts on X, and $x \in X$, then $\{g \cdot x \mid g \in G\} = Gx$ is the *orbit* of x under G, and

$G \supset G_x = \{g \in G \mid gx = x\}$ is the *isotropy* subgroup, or *stabilizer*, of x.

The map

$$\begin{tikzcd} \mapsto gx \ : \ G \arrow[r] \arrow[dr] & Gx \\ & G/G_x \arrow[u, "\cong"'] \end{tikzcd}$$

factors through a bijection of the *orbit* with the *quotient* of G by the isotropy group.

Ex. If G (finite) acts linearly on a vector space V, then G acts on $V^+ = S(V)$

Ex. Let $\mathbb{T} = \{ \ \in \mathbb{C} \mid \ = 1\}$ be the circle group, acting on $\mathbb{C}$ by

$$\mathbb{T} \times \mathbb{C} \ni (u, z) \mapsto uz \in \mathbb{C}.$$

This extends to an action on $\mathbb{C}_+ \cong S^2$:

← orbit of $1 \in \mathbb{C}$

In this case 0 and ∞ have $\mathbb{T}$ as their isotropy groups, and all other points have $\{1\} \subset \mathbb{T}$ as isotropy group.

A group action is *free* if all its isotropy groups are trivial (= the one-element group).

Ex The action of $\Sigma(n)$ on $\mathrm{Config}^n(\mathbb{C})$ is free.

a *finite* group

Claim If G acts *freely* on a manifold, then the quotient space X/G is a manifol

Proof This takes for granted the fact that a group action defines an equivalence relation (ie $x_0 \sim x_1 \Leftrightarrow \exists g \in G$ such that $gx_0 = x_1$); the quotient space X/G is the quotient of X by that equivalence relation.

I will also assume that X *has* a metric; then

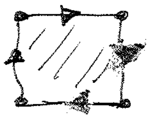
$$X \ni x \mapsto \underbrace{\min_{g \in G - \{e\}} \{\text{dist}(x, gx)\}}_{:= \delta(x)} \in \mathbb{R}$$

is continuous (since $\# G < \infty$) and positive ($\delta(x) > 0$: otherwise $\exists\, x$ with $gx = x$ for some $g \in G - \{e\}$, ie $G_x \neq \{e\}$, which is impossible since the action is free.

Now if $\delta(x) > \epsilon > 0$ then $[x] \in X/G$ has a neighborhood iso to an ϵ-ball around x.

Ex $\text{Config}^n(\mathbb{C})/\Sigma(n)$ is the manifold of *unordered* n-tuples of *distinct* points in the plane.

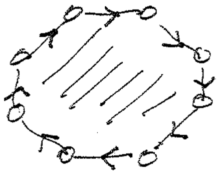
Ex the group $Sl_2(\mathbb{Z})$ of 2×2 integral matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with determinant $ad - bc = 1$ acts on the upper half-plane $= \{z \in \mathbb{C} \mid \text{Im}\, z > 0\}$ by $z \mapsto \frac{az+b}{cz+d}$. Is this action free?

Ex The torus $T^2 = T \times T \cong \mathbb{R}^2/\mathbb{Z}^2$ can also be defined as the quotient of the unit square $[0,1] \times [0,1]$ by the equivalence relation  which identifies opposite sides.

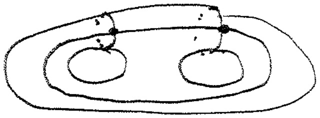
More generally the equivalence relation

Notation If $X, Y \in G$-spaces, then $X \times Y$ is a G-space by $g(x,y) = (gx, gy)$ (the 'diagonal' action). We sometimes write $X \times_G Y$ for the quotient $(X \times Y)/G$, eg $X \times_G pt = X/G$. NB this is *not* a fiber product!

xxiii)



which glues together opposite sides of a 4g-gon defines a surface



with g "holes".

Ex

our old friend the Möbius strip is a manifold with boundary.

glueing a disk along its boundary defines a real projective plane : alternately,

consider a 2-sphere with a "window" and glue the window's edges together ...

② 2.7 Projective Spaces

$\mathbb{RP}^n$ is the space of real lines through the origin in (n+1)-dimensional real Euclidean space $\mathbb{R}^{n+1}$; similarly, *complex* projective

xxiv)

space is the analogous construction over the complex numbers.

[These are significant ; primary examples of *moduli spaces* : topological spaces whose points correspond to interesting objects with internal structure : lines, planes, etc in Euclidean spaces, configuration spaces (ordered or not) ...]

Definition $\mathbb{R}^\times \cong \{\pm 1\} \times \mathbb{R}$ (via the logarithm

$$\mathbb{C}^\times \cong \mathbb{T} \times \mathbb{R}$$

are the groups of *nonzero* real & cx numbers, under multiplication. These are group actions

$$\mathbb{R}^\times \times (\mathbb{R}^{n+1} - 0) \longrightarrow (\mathbb{R}^{n+1} - 0)$$
$$r, (x_0, \dots, x_n) \longmapsto (rx_0, \dots, rx_n)$$

$$\mathbb{C}^\times \times (\mathbb{C}^{n+1} - 0) \longrightarrow (\mathbb{C}^{n+1} - 0)$$
$$u, (z_0, \dots, z_n) \longmapsto (uz_0, \dots, uz_n)$$

Claim These are *free* actions.

Proof We have to show there are no nontrivial isotropy groups; so suppose

$$u(z_0,\dots,z_n) = (uz_0,\dots,uz_n) = (z_0,\dots,z_n)$$

for $u \neq 1$ in $\mathbb{C}$. Not all of the z_i can be zero, so there is some $z_k \neq 0$, such that $uz_k = z_k$, hence $u = 1$, a *contradiction*.

Definition

$$\mathbb{CP}^n := (\mathbb{C}^{n+1}-0)/\mathbb{C}^\times$$

$$\mathbb{RP}^n := (\mathbb{R}^{n+1}-0)/\mathbb{R}^\times$$

are the quotient spaces associated to these actions.

They are *manifolds*. We write $[z_0:\dots:z_n]$ for the eq. class of $\underline{z} \in \mathbb{C}^{n+1}$; similarly, for $\underline{x} \in \mathbb{R}^{n+1}$.

Proof There are alternate descriptions of these spaces which are sometimes useful:

$$\exists \quad S^{2n-1}/\mathbb{T} \xrightarrow{\cong} (\mathbb{C}^{n+1}-0)/\mathbb{C}^\times$$

$$S^n/\{\pm 1\} \longrightarrow \mathbb{RP}^n$$

Since ±1 is a finite group acting freely, $\mathbb{RP}^n$ is a manifold by a previous argument; but $\mathbb{T}$ is infinite, so we'll just construct a covering of $\mathbb{CP}^n$ by coordinate patches,

ie $$\mathbb{C}^n \ni (z_1,\dots,z_n) \mapsto [z_1:\dots:z_{k-1}:\underset{\uparrow}{1}:z_k:\dots:z \in \mathbb{CP}^n$$

insert 1 in k^{th} place

Ex. $$\mathbb{CP}^1 \ni [z_0:z_1] \longrightarrow z_0 z_1^{-1} \in \mathbb{C}_+ = \mathbb{C} \cup \infty \cong S^2$$

sends $[0:1]$ to 0,
$[1:0]$ to ∞.

Ex The *Hopf map*

$$\begin{tikzcd} S^3 = \{(z_1,z_2) \in \mathbb{C}^2 \mid |z_1|^2+|z_2|^2 = 1\} \arrow[d, "\eta"] \\ S^2 = \{[z_1:z_2] \in \mathbb{CP}^1\} \end{tikzcd}$$

necessarily $(z_1,z_2) \neq (0,0)$!

has $0 \times \mathbb{T} \subset S^3$ as "fibre" $\eta^{-1}([0:1]) = \eta^{-1}(0$

Ex. Let $p(\underline{z}) \in \mathbb{C}[z_1,\dots,z_n]$ be a polynomial of degree d in n variables; then

$$P(z_0,\dots,z_n) := z_0^d\, p(z_1/z_0,\dots,z_n/z_0) \in \mathbb{C}[z_0,\dots,z_n]$$

is a *homogeneous* polynomial, of degree d in n+1 variables; eg

$p(x,y) = y^2 - (x^3+ax+b)$ has $d=3$, $n=2$

$P(X,Y,Z) = Y^2Z - (X^3+aXZ^2+bZ^3)$

with $x = XZ^{-1}$, $y = YZ^{-1}$.

Claim the "hypersurface of degree d"

$$Z(P) := \{[z_0:\dots:z_n] \in \mathbb{CP}^n \mid P(z_0,\dots,z_n)=0\}$$

is *generically* a smooth $2(n-1)$-dimensional submanifold.

Proof 0 is *generically* a regular value of p,

by Sard & Thom, so $p^{-1}(0)$ is then a manifold, by the implicit function theorem.

A classical theorem of algebraic geometry asserts that a polynomial $p \in \mathbb{C}[x,y]$ of degree d defines a smooth 2-dimensional manifold $Z(p) \subset \mathbb{CP}^2$ with $g = \frac{1}{2}(d-1)(d-2)$ "holes", eg

$$y^2 = x^3 + ax + b,$$

for generic a, b, defines a two-torus $Z(p) = T^2 \subset \mathbb{CP}^2$.

㉘ Associated bundles and differential forms

Definition A *frame* $(\underline{v}_1, \ldots, \underline{v}_n)$ on an n-dim'l real vector space V is an ordered set of linearly independent elements. Any $\underline{x} \in V$ can thus be uniquely expressed as a sum $\underline{x} = \sum_{i=1}^{i=n} x_i \underline{v}_i$. The set $F(V)$ of frames on V has an action

$$Gl_n(\mathbb{R}) \times F(V) \longrightarrow F(V)$$

$$T = (T_{ik}), (\underline{v}_k) \longmapsto (T_i(\underline{v}) = [\sum_{k=1}^{k=n} T_{ik} \underline{v}_k)$$

of the group $Gl_n(\mathbb{R})$ of invertible real matrices. Elementary linear algebra says that this action is *transitive*: for any two frames $(\underline{v})$ and $(\underline{v}')$ on V there is a (unique!) $T \in Gl_n(\mathbb{R})$ such that $T(\underline{v}) = (\underline{v}')$. Thus the orbit map

$$Gl_n(\mathbb{R}) \ni T \overset{\cong}{\longmapsto} T(\underline{v}) \in F(V)$$

is an isomorphism of sets, which makes $F(V)$ into a space homeomorphic (but not uniquely so!) to $Gl_n(\mathbb{R})$.

If M is a manifold,

$$F(M) = \coprod_{x \in M} F(T_x M) \longrightarrow M$$

defines the (principle) *frame bundle* of M. There is a *free* action

$$Gl_n(\mathbb{R}) \times F(M) \longrightarrow F(M)$$

with M as its quotient space.

If $\rho : Gl_n(\mathbb{R}) \longrightarrow \mathrm{Aut}(V)$ is a *representation* of $Gl_n(\mathbb{R})$ on a vector space V, ie a family

$$\rho(T) : V \longrightarrow V$$

of *linear* maps satisfying $\rho(T_0 \cdot T_1) = \rho(T_0) \cdot \rho(T_1)$, then the quotient

$$(F(M) \times V)/Gl_n(\mathbb{R}) \; := F(M) \times_{Gl_n(\mathbb{R})} V \longrightarrow M$$

defined by the *diagonal* action

$$Gl_n(\mathbb{R}) \ni T, \; (\underline{v}, w) \longmapsto (T(\underline{v}), \rho(T) w) \in F(M) \times$$

[*warning*: this is *not* a fiber product; unfortunately these two conflicting notations are well-established] is a bundle with fibre V over M, called the vector bundle *associated* to the tangent bundle, defined by the representation ρ.

Ex. The dual $(\mathbb{R}^n)^*$ of $\mathbb{R}^n$ is a representation.

Its associated vector bundle is the <u>cotangent</u> bundle $\coprod_{x \in M} T^*_x M \to M$, (where $T^*_x M = \mathrm{Hom}_{\mathbb{R}}(T_x M, \mathbb{R})$ is the vector space dual to the tangent space). Elements of the tangent space are sometimes denoted $\partial_i = \partial/\partial x_i$, and the dual elements of the cotangent space are then denoted dx_i.

More generally the exterior powers $\Lambda^k(\mathbb{R}^n)$ and their duals $\Lambda^k(\mathbb{R}^{n*})$ define bundles $\Lambda^k(TM)$, $\Lambda^k(T^*M)$ etc.

A <u>section</u> s of a vector bundle $E \xrightarrow{\pi} X$ is an element of the vector space

$$\{s : X \to E \mid \pi \circ s = \mathrm{id}_X\}_* := \Gamma_X(E)$$

ie if $x \in X$ then $s(x) \in E_x = \pi^{-1}(x) \subset E$. [Since E_x is a vector space, the sum $s_0 + s_1$ of two sections is again a section:

$s_0(x), s_1(x) \in E_x$ so $s_0(x) + s_1(x) \in E_x$.]

Sections of the tangent bundle of a manifold are called vector fields, and sections of the cotangent bundle are called one-forms; the traditional notation is

$$\Omega^k(M) = \Gamma_M(\Lambda^k(T^*(M))).$$

<u>Ex</u> If $f : M \to \mathbb{R}$ is a smooth function, its tangent map $Tf : T(M) \to T(\mathbb{R})$ defines a section $x \mapsto (T_x f) \in \mathrm{Hom}(T_x M, \mathbb{R}) = T^*_x M$ called $df \in \Omega^1(M)$.

Sections of $\Lambda^k(T^*(M))$ are usually written in local coordinates as expressions of the form

$$\Omega^k(M) \ni \alpha = \sum \alpha_I(x)\, dx_I ,$$

where $I = i_1 < \dots < i_k$ with entries from the set $\{1, \dots, n\}$, and $dx_I := dx_{i_1} \wedge \dots \wedge dx_{i_k} \in \Lambda^k(T^*_x M)$.

The product map $\underset{e_{i_1}\wedge\dots\wedge e_{i_k},\, e_{j_1}\wedge\dots e_{j_l} \mapsto e_{i_1}\wedge\dots\wedge e_{j_l}}{\Lambda^k V \times \Lambda^l V \to \Lambda^{k+l}(V)}$ makes the sum $\bigoplus_{k \geq 0} \Omega^k(M) = \Omega^*(M)$ into an <u>algebra</u> over the ring $C^\infty(M, \mathbb{R})$ of smooth functions on M. $(= \Omega^0(M))$

It is *not* commutative: if $\alpha \in \Omega^k$ and $\beta \in \Omega^l$, then $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha \in \Omega^{k+l}$.

There is a differential operator

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

defined locally by

$$d\alpha = d\left(\Sigma_I \, \alpha_I(x) \, dx_I\right)$$

$$:= \sum_{i,I} \frac{\partial \alpha_I(x)}{\partial x_i} dx_i \wedge dx_I .$$

It satisfies a *Leibniz formula* ($\alpha \in \Omega^k$, $\beta \in \Omega^l$)

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

as well (by the equality of mixed partial derivatives) as the equation

$$d(d\alpha) = 0.$$

$\boxed{\text{The de Rham complex}}$

This defines a sequence

$n = \dim M$

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \to \Omega^2 \to \cdots \to \Omega^n \to 0$$

of vector spaces and homomorphisms which is *not quite exact*: image $d \subset$ ker d because $d \circ d = 0$, but in general the quotient

$$H^k_{dR}(M) := \ker d_k / \operatorname{image} d_{k-1} \neq 0 .$$

Deep work in *Analysis* shows that if M is a *compact* manifold, these (a *priori* infinite-dimensional) quotient spaces are *finite* dimensional. They can be interpreted as a sophisticated version of theorems about existence and uniqueness of solutions of systems of some partial differential equations.

Finally: if $F : M' \to M$ is a smooth map of manifolds,

$$F^* : \Omega^*(M) \to \Omega^*(M'),$$

$$F^*(\alpha) = F^*\left(\Sigma \, \alpha_I(x') \, dx'_I\right)$$

$$:= \Sigma \, (\alpha_I \circ F)(x) \, F^*(dx'_I),$$

Jacobian matrix

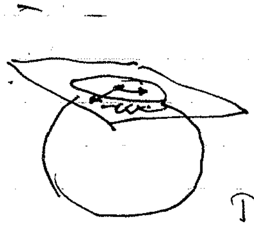
$$F^*(dx_i) = \sum_{k=1}^{k=n} \frac{\partial x_i}{\partial x'_k} dx'_k = \sum_k F'(x')_{ik} \, dx'_k$$

makes the de Rham complex a *contravariant* functor:

$$d_{M'} F^* = F^* d_M .$$

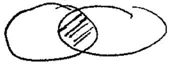
viii*)

Digression A Riemannian metric on a the tangent bundle of manifold M is an isomorphism $TM \to T^\vee M$ of vector bundles, such that $T_xM \to T_x^\vee M = \mathrm{Hom}(T_xM, \mathbb{R})$ is a Riemannian metric on the vector space T_xM (as in §1.4 p xix). (Existence of such metrics can be shown using *partitions* of *unity* ...). A choice of Riemannian metric defines an *exponential* map $T_xM \xrightarrow{\exp}$ (nbd of x in M)



which sends a tangent vector at x to a *geodesic* in M with that tangent vector as its initial condition.

This requires some ODE theory which I will omit. A consequence is that the image of a small open ball in the tangent space is a *geodesically convex* open neighborhood of x. A corollary is the existence of a 'good covering' of M, by geo. convex opens, such that the intersection of any two such is *also* geodesically convex. This is useful later in the proof of *Poincaré Duality*.



xxiii*)

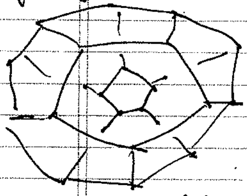
An *orbifold* is a useful generalization of a manifold; a space with a coordinate atlas defined by maps $\varphi_i : U_i/G_i \to X$, where U_i is open in Euclidean space, and G_i is a finite group (acting linearly and faithfully ?) on U_i.

The point of this addendum is to note a useful idea of Kawasaki: that an orbifold has a 'frame bundle', which is a space with an action of $Gl_n(\mathbb{R})$, such that the isotropy group of any point is finite.

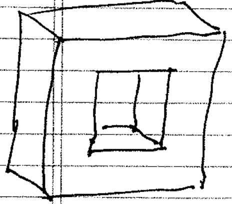
(some afterthoughts)

§ Starting with Euler

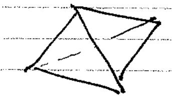
(3.1) In this talk I'll assume some familiarity with the basics of set and category theory, vector spaces or abelian groups, etc.; but I will not assume a lot of non-intuitive knowledge about topological spaces. Instead I'll follow Riemann, and talk at first about polyhedra (a.k.a. simplicial complexes)



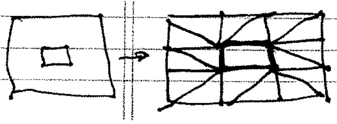
which are familiar and intuitive — and are all around us, eg as the basis for wireframe animation in video games



In three dimensions polyhedra are defined by their vertices, edges, and faces:

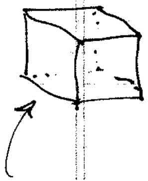


and it will be easiest if we think of our polyhedra as built up of simplices, ie as triangulated complexes



And I want to start with a definition / theorem going back to Euler, who observed? proved? realized? that any two-dimensional polyhedron with the topology of a two-sphere

eg

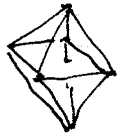


satisfied Euler's relation

$$F - E + V = 2$$

$$\boxed{\#\text{ of faces} - \#\text{ of edges} + \#\text{ of vertices} = 2}$$

6 faces
12 edges
8 vertices



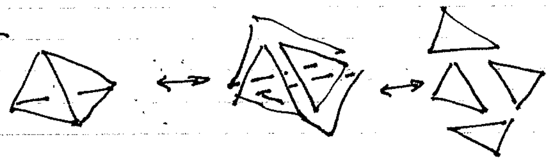
8 faces
6 vertices
12 edges

It was eventually realized that this is a fact of some depth, for example because it leads to a remarkably simple proof of the classification of the Platonic solids, ie polyhedra such that all faces have the same number of edges (ie they are all n-gons) (≥ 3) and all vertices have the same number (call it $k (>2)$) of incoming edges. If we blow the polyhedron up, ie think of it as obtained from its faces glued

Part I : from Euler to Noether

iii).

together



then each face has n edges, so if there are F faces E edges then $nF = 2E$.

Similarly at each of the V vertices there are k incident "half-lines"

so $kV = 2E$. But then

$$2 = F - E + V = \frac{2}{n}E - E + \frac{2}{k}E$$

hence

$$\frac{1}{n} + \frac{1}{k} = \frac{1}{2} + \frac{1}{E}.$$

But $E > 0$ so $\frac{1}{n} + \frac{1}{k} > \frac{1}{2}$: in particular both n and k can't be very big, and in fact there are only five possibilities:

$(n,k) = (3,3), (3,4), (4,3), (5,3), (3,5)$

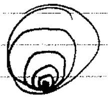
tetrahedron, octahedron, cube, dodecahedron, icosahedron

---

iv)

Since the 13 books of Euclid's elements are (said to be) organized to prove exactly this result, this argues for the depth of Euler's theorem.

We will see [Rotman Ch 8 p 221] in this course that the Euler characteristic $\chi(X) \in \mathbb{Z}$ makes sense for a 'reasonable' space —

[there are plenty of 'unreasonable' spaces, eg the Hawaiian earring  , for which $\chi = -\infty$]

and that for such spaces, it satisfies the remarkable identities

$$\boxed{\begin{aligned} \chi(U \cup V) + \chi(U \cap V) &= \chi(U) + \chi(V) \\ \chi(U \times V) &= \chi(U) \cdot \chi(V) \end{aligned}}$$

(which should be familiar from measure theory).

Moreover, χ is a topological — and even <u>homotopy</u> - <u>theoretic</u> [Rotman Ch 1 p 16] invariant.

At this point it will be useful to make some use of the language of categories, to distinguish some properties of *spaces* from *sets*: in particular, sets are pretty sparse when it comes to their classification. A set S has essentially only one set-theoretic invariant, its *cardinality* #(S) = the number of its elements: which satisfies the identities (if #(S) < ∞ !) [verifiable by looking at Venn diagrams!]

$$\chi(S \cap T) + \chi(S \cup T) = \chi(S) + \chi(T),$$

$$\chi(S \times T) = \chi(S) \cdot \chi(T),$$

and $\chi(S) \geq 0$.

The Euler characteristic satisfies

$$\boxed{\chi(F) = \#(F)}$$

if F is a *finite set* (regarded as a very simple kind of topological space), but in general it can be negative.

Remarkably, then,

> The Euler characteristic defines a new, and unfamiliar extension of the notion of number of elements in a set: it represents a *new kind* of book-keeping, which somehow takes into account the *geometry* of the space, generalizing our basic intuitive notions coming from counting.

At this point it is worth working out some elementary examples. It follows from *homotopy-invariance* (which I'll discuss in more detail in the next class) that

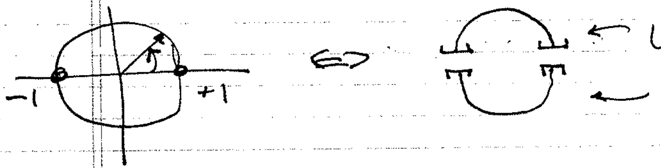
$$\chi(\text{point}) = \chi(n\text{-ball}) = 1$$

where an n-ball is a space of the form

$$\{\underline{x} \in \mathbb{R}^n \mid |x| \leq R\};$$

in particular, it is *contractible* and so is in some sense just a 'thickening' of a point. If n=1 we see that $\chi(\text{closed interval } [a,b]) = 1$, and if we decompose a circle as

vii) $\mathbb{T} = \{u \in \mathbb{C} \mid |u| = 1\}$, $u = \exp(2\pi i t)$, $t \in \mathbb{R}/\mathbb{Z}$ $\cong [0,\pi]$



$U = \{\exp(2\pi i t),\ 0 \le t \le \pi\}$

$V = \{\exp(2\pi i t),\ \pi \le t \le 2\pi\} \cong [\pi, 2\pi]$

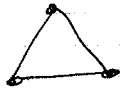
$\mathbb{T} = U \cup V$, $U \cap V = \{\pm 1\}$,

Then

$$\chi(\mathbb{T}) + \underset{=2}{\chi(\pm 1)} = \underset{=1}{\chi(I_+)} + \underset{=1}{\chi(I_-)}$$

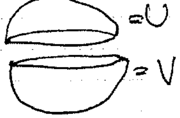
$\Rightarrow$ $\boxed{\chi(\mathbb{T}) = 0}$

$\mathbb{T}$ = the 1-torus; $\mathbb{T}^2 \cong \mathbb{T} \times \mathbb{T}$  is the 2-torus –

[In the language of polyhedra, we can think of the circle $\mathbb{T}$ as the boundary  of a triangle, ie as a "simplicial complex" with 3 vertices, 3 edges, and no faces, so

$\chi(\mathbb{T}) = 0 - 3 + 3 = 0$

is consistent with this calculation.]

Let's proceed to higher-dimensional calculations. We can present the 2-sphere as the union of two disks  $= U$, $= V$ intersecting along $U \cap V \cong$ a circle:

viii)

In this case

$S^2 = U \cup V$, $U \cap V = S^1$, and U, V are both contractible; so

$$\chi(S^2) + \overset{0}{\overset{\|}{\chi(S^1)}} = \chi(pt) + \chi(pt) = 2,$$

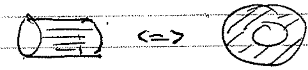
ie $\boxed{\chi(S^2) = 2}$.

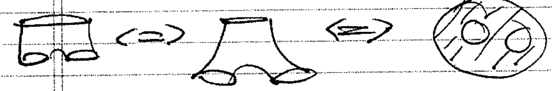
Proceeding by induction, it follows that

$$\chi(S^n) = 1 + (-1)^n = 2 \text{ if } n \text{ is even}$$
$$= 0 \text{ if } n \text{ is odd}$$

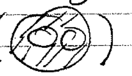
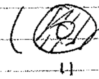
In another direction, the product formula implies

$$\chi(\mathbb{T} \times I) = \chi(\mathbb{T}) \cdot \chi(I) = 0 \cdot 1 = 0,$$

ie that the Euler characteristic of the cylinder  the annulus is zero, and hence that the Euler characteristic of the 'pair of pants' or 'knickers' space



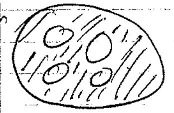
the 'vertex operator' space V in string theory

satisfies $\underset{V}{\chi}\big($  $\big) + \overset{1}{\overset{\|}{\chi}}\big($  $\big)_{U} = \overset{U \cup V}{\chi}\big($  $\big)_{\overset{\|}{0}} + \overset{U \cap V}{\chi}\big(\ \big)_{\overset{\|}{0}}$

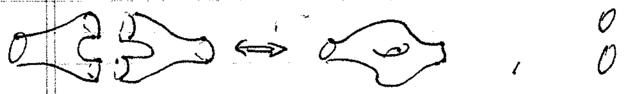
ie $\boxed{\chi(V) = -1}$.

ix)

This generalizes, to say that the disk with n holes has Euler characteristic

$\chi($  $) = 1-n$

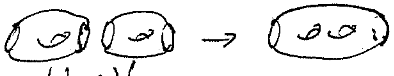
We can now do more advanced calculations, such as



$U \cup V$, $U \cap V = T \sqcup T$

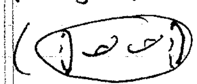
ie $\chi(U \cup V) + \chi(\overset{\circ}{\overset{\shortparallel}{U \cap V}}) = \underset{-1}{\chi(U)} + \underset{-1}{\chi(U)}$,

so $\chi($  $) = -2$,
↖ two-holed torus

hence  has $\chi = -4$.
$U \cup V$

(with $\chi(U \cap V) = \chi(\text{circle}) = 0$)

Capping off the two holes implies

$\chi($  $) = -2 = 2(1-g)$, $g = 2$:

by induction, a surface  with g "holes", ie of "genus g"

has $\boxed{\chi = 2(1-g)}$

x)

That the Euler characteristic can be negative has significant geometric consequences: for example the Gauss-Bonnet theorem asserts that the integral of the Gaussian curvature K
closed S
over a surface with Riemannian metric is

$$\int_S K\, d\sigma = 2\pi\, \chi(S) \;;$$

Thus a surface of genus $g > 1$ is in some sense 'mostly' negatively curved. String theorists would have us believe that the Euler characteristic of our very own Universe has absolute value 3 (because there are three kinds — e, μ, τ — of neutrinos, duh!); but because of something called mirror symmetry, they don't know if its sign is positive or negative.

xi)

3.2 Noether's *categorification* of the Euler characteristic

The notion of a topological *space* is an enormous enrichment of the notion of a *set*; one way to see this is to realize that maps between spaces encode *much* more information maps between sets.

E. Noether is credited with seeing that the Euler characteristic is a kind of shadow of a deeper invariant, that it should be regarded as the dimension of (*something like*) a vector space.

Rotman Ch0 p3

Definition A graded vector space $V_* = \bigoplus_{k\in\mathbb{Z}} V_k$ is a family of vector spaces indexed by the integers. [In many examples $V_k = \{0\}$ if $k<0$, and we can take the indexing set to be $\mathbb{N}$.] Graded vector spaces form a category,

$$\mathrm{Hom}_*(V,W) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_k(V_*,W_*)$$

xii)

alternately: $\phi : V_* \to W_{*+k}$

is itself a graded vector space: a homomorphism ($\phi \in \mathrm{Hom}_k(V_*,W_*)$) 'of degree k' is an element of the vector space

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(V_i, W_{i+k})\ ;$$

such a ϕ is thus itself a family $\phi = \bigoplus \phi(i)$

$$\phi(i) : V_i \longrightarrow W_{i+k}$$

of linear transformations, such that if

$$\phi \in \mathrm{Hom}_k(V_*,W_*),\quad \psi \in \mathrm{Hom}_l(W_*,Z_*)$$

then

$$\psi(i+k)\circ\phi(i) := (\psi\circ\phi)(i) : V_i \xrightarrow{\phi(i)} W_{i+k} \xrightarrow{\psi(i+k)} Z_{i+(k+l)}$$

defines an element of $\mathrm{Hom}_{k+l}(V_*,W_*)$.

[This is a technical mechanism for dealing with the fact that topological spaces are themselves implicitly graded by *dimension*; but dealing with this is tricky, because dimension is *not* a homotopy-invariant notion.]

Similarly $V_*\otimes W_*$ is the graded vector space with $\bigoplus_{i+j=k} V_i\otimes W$ in degree k.

xiii)

Remark There are completely analogous categories of graded *abelian groups* and *graded modules* over a ring R; and we will eventually work systematically in such categories, but for now, graded vector spaces will suffice.

We will also have to worry about finiteness conditions: for example there is a category of graded *finite-dimensional vector spaces*, and one of graded *finitely-generated abelian groups*. [Recall that a finitely-generated abelian group

$$A \cong \mathbb{Z}^r \oplus F$$

for some integer $r \geq 0$ called its rank, with F a *finite* abelian group, and that

$$A \otimes \mathbb{Q} \cong \mathbb{Q}^r$$

(with $\mathbb{Q} \subset \mathbb{R}$ the field of rational numbers). In fact this defines a *functor* $A \mapsto A \otimes \mathbb{Q}$ from (f.g. Abs) to (f.d $\mathbb{Q}$-Vector Spaces), which sends the *rank* to the *dimension*.

xiv)

Definition If $V_* = \oplus V_i$ is finite-dimensional (as an *ungraded* F-vector space), its *graded dimension* is the *alternating* sum

$$\text{gr dim } V_* := \sum_{k \in \mathbb{Z}} (-1)^k \dim_F V_k \in \mathbb{Z}$$

of the dimensions of its components. Note that it can be *negative*!

If V_* and W_* are graded vector spaces, their *direct sum* is the graded vector space with

$$(V \oplus W)_k = V_k \oplus W_k \ ;$$

evidently

$$\boxed{\text{gr dim } (V_* \oplus W_*) = \text{gr dim } V_* + \text{gr dim } W_*}$$

There is also a notion of *tensor product* for graded vector spaces:

$$(V \otimes W)_i := \bigoplus_{j+k=i} V_j \otimes W_k$$

To see that

$$\boxed{\text{gr dim } (V \otimes W)_* = (\text{gr dim } V_*) \cdot (\text{gr dim } W_*)}$$

define the Hilbert polynomial

$$p_V(t) = \sum_k (\dim V_k) \cdot t^k \in \mathbb{Z}[t, t^{-1}]$$

(of a finite-dimensional graded vector space).

It is then easy to see that

$$\text{gr dim } V_* = p_V(-1)$$

and that

$$p_{V \otimes W}(t) = p_V(t) \cdot p_W(t).$$

<u>RECALL</u> that a sequence

$$\cdots V' \xrightarrow{\alpha} V \xrightarrow{\beta} V'' \longrightarrow \cdots$$

of R-modules and homomorphisms is <u>exact</u> <u>at</u> <u>V</u>
(abelian grps, vector spaces)

$$\Leftrightarrow \text{image of } \alpha = \{\alpha(v') \in V \mid v' \in V'\}$$
$$=$$
$$\text{kernel of } \beta = \{v \in V \mid \beta(v) = 0\}.$$

<u>Example</u> If V, W are abelian groups, then

$$0 \longrightarrow V \xrightarrow{\alpha} W \xrightarrow{\beta} Z \longrightarrow 0$$

is exact iff α is one-to-one, and $Z \cong W/\text{image } \alpha$;

Moreover, if any two of V, W, Z are finite-dimensional vector spaces, then

$$\dim V - \dim W + \dim Z = 0,$$

eg $V \cong \mathbb{R}^n$, $Z \cong \mathbb{R}^m \Rightarrow W \cong \mathbb{R}^{n+m}$.

<u>Note</u> $0 \to A \xrightarrow{\alpha} B \to 0$ is exact $\Leftrightarrow \alpha : A \cong B$ is an <u>iso</u>.

Any $\alpha : V \to W$ has a factorization

$$\begin{tikzcd}
V \arrow[rr, "\alpha"] \arrow[dr] & & W \\
& \operatorname{coker}\alpha := V/\ker\alpha \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & 0
\end{tikzcd}$$

with the diagonal sequences exact, (& of finite length?)

and any <u>long</u> exact sequence can thus be presented as an <u>iterated</u> <u>splice</u>

$$\begin{tikzcd}[column sep=small]
& & & & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & \\
& & & & & V_2/\ker\alpha_2 \arrow[ur] \arrow[dr] & & & & V_4/\ker\alpha_4 \arrow[ur] \arrow[dr] & & & \\
0 \arrow[r] & V_0 \arrow[r, "\alpha_0"] & V_1 \arrow[rr, "\alpha_1"] \arrow[dr] & & V_2 \arrow[rr, "\alpha_2"] \arrow[ur] & & V_3 \arrow[rr, "\alpha_3"] \arrow[dr] & & V_4 \arrow[rr, "\alpha_4"] \arrow[ur] & & V_5 \arrow[rr] \arrow[dr] & & 0 \\
& & \left(\begin{smallmatrix} \operatorname{im}\alpha_1 \\ = \ker\alpha_2 \end{smallmatrix}\right) = & V_1/V_0 \arrow[ur] \arrow[dr] & & & & V_3/\left(\begin{smallmatrix} \operatorname{im}\alpha_2 = \\ \ker\alpha_3 \end{smallmatrix}\right) \arrow[ur] \arrow[dr] & & & & V_5/(\ker\alpha_5 = V & \\
& & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & & & &
\end{tikzcd}$$

of (diagonal) <u>short</u> exact sequences

$$\boxed{\begin{array}{l} \underline{\text{Exercise}}: \text{ If the } V_i \text{ are finite-dimensional vector spaces,} \\ \text{then } \sum (-1)^\ell \dim V_\ell = 0. \end{array}}$$

<u>Corollary</u> (a notational variant) If

$$\cdots \longrightarrow A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \xrightarrow{\gamma_i} A_{i-1} \xrightarrow{\alpha_{i-1}} B_{i-1} \xrightarrow{\beta_{i-1}} C_{i-1} \longrightarrow \cdots$$

is a long exact sequence of f.dim'l vector spaces (ie

$\alpha_*: A_* \to B_*$ and $\beta_*: B_* \to C_*$ are homomorphisms of graded vector spaces of degree <u>zero</u>, and $\gamma_*: C_* \to A_{*-1}$ is of <u>degree -1</u>, then

$$\operatorname{gr\,dim} A_* - \operatorname{gr\,dim} B_* + \operatorname{gr\,dim} C_* = 0 .$$

<u>Remark</u> To save space, such a long exact sequence can be rolled up into an <u>exact</u> triangle

$$\begin{tikzcd} A_* \arrow[rr, "\alpha_*"] & & B_* \arrow[dl, "\beta_*"] \\ & C_* \arrow[ul] & \end{tikzcd}$$

$(\deg \gamma_* = -1)$.

<u>NOETHER's</u> <u>INSIGHT</u> is that (under reasonable finiteness hypotheses) the Euler characteristic

$$\boxed{\chi(X) = \operatorname{gr\,dim} H_*(X, \mathbb{Q})}$$

of a topological space is the (graded) dimension of a certain (graded) vector space (called the <u>rational homology</u> of the space, <u>functorially</u> associated to the space: that is, a map $f: X \to Y$ defines a linear transformation $f_* (= H(f)): H_*(X, \mathbb{Q}) \to H_*(Y, \mathbb{Q})$

satisfying (for a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$ of maps) the relations $(g \circ f)_* = g_* \circ f_*$, $\mathrm{id}_* = \mathrm{id}$.

Moreover, the '<u>counting measure</u>' property

$$\chi(U \cup V) + \chi(U \cap V) = \chi(U) + \chi(V)$$

is the consequence of the existence of an <u>exact</u> <u>triangle</u>

$$\begin{tikzcd} H_*(U \cap V) \arrow[rr, "\alpha_*"] & & H_*(U) \oplus H_*(V) \arrow[dl, "\beta_*"] \\ & H_*(U \cup V) \arrow[ul, "\gamma_*"] & \end{tikzcd}$$

associated to the fibre product diagram

$$\begin{tikzcd} & U \cap V \arrow[dl] \arrow[dr] & \\ U \arrow[dr] & & V \arrow[dl] \\ & U \cup V & \end{tikzcd}$$

and the <u>product</u> <u>formula</u> (i.e. ('Fubini's Theorem') is the consequence of an isomorphism

$$H_*(X \times Y, \mathbb{Q}) \cong H_*(X, \mathbb{Q}) \otimes H_*(Y, \mathbb{Q})$$

of graded vector spaces.

---

§33

Rotman Ch 9 p 230

The properties listed above are very close to the <u>Eilenberg</u>-Steenrod <u>axioms</u> <u>for</u> a <u>homology</u> <u>theory</u>. They can be made more precise.

xix)

In particular, there is a refinement of these more precise vector spaces, to integral homology groups $H_*(X,\mathbb{Z})$ such that

$$H_*(X,\mathbb{Z}) \otimes \mathbb{Q} \cong H_*(X,\mathbb{Q}).$$

Similarly, the induced homomorphisms on these groups satisfy the Homotopy Axioms :

$\boxed{\text{Rotman Ch I} \; p14-19}$

if $f \simeq g : X \to Y$ are homotopic maps, then $f_* = g_* : H_*(X,\mathbb{Z}) \to H_*(Y,\mathbb{Z})$.

Here are a few basic examples :

1) Euclidean n-balls (open or closed) are homotopy-equivalent to a point, so

$$H_*(B_n,\mathbb{Z}) \cong H_*(pt,\mathbb{Z}) = \mathbb{Z} \quad \text{if } * = 0$$
$$= 0 \quad \text{otherwise}$$

2) The homology of the n-sphere $S^n = \partial B^{n+1}$ is

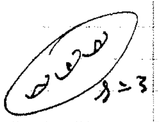
$$H_*(S^n,\mathbb{Z}) = \mathbb{Z} \quad \text{if } * = 0 \text{ or } n$$
$$= 0 \quad \text{otherwise,}$$

provided $n > 0$: but $S^0 = \partial B^1$ is a finite set with two elements, so

$$H_*(S^0,\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \quad \text{if } * = 0$$
$$= 0 \quad \text{otherwise.}$$

xx)

3) If F_g is a smooth surface with g "holes" then

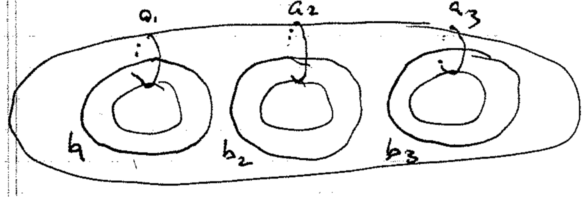


$$H_*(F_g,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & * \geq 2 \\ \mathbb{Z}^{2g} & * = 1 \\ \mathbb{Z} & * = 0 \end{cases}$$

eg when $g = 1$, F is the torus T^2, so

$$H_*(T^2,\mathbb{Z}) = H_*(S^1,\mathbb{Z}) \otimes H_*(S^1,\mathbb{Z}) = \begin{cases} \mathbb{Z} & *= \\ \mathbb{Z}^2 & *= \\ \mathbb{Z} & * \end{cases}$$

We will see later that classes in $H_k(X,\mathbb{Z})$ can be interpreted as (equivalence classes of) geometric 'cycles' in X. For example, $H_1(F_g,\mathbb{Z})$ has a standard basis of cycles $a_1,\ldots,a_g; b_1,\ldots,b_g$, usually drawn as



4) $$H_*(\mathbb{CP}^n,\mathbb{Z}) \cong \mathbb{Z} \quad \text{if } 0 \leq * \leq 2n \text{ is even}$$
$$= 0 \quad \text{otherwise}$$

(consistent with $\mathbb{CP}^1 \cong S^2$).

$$H_*(\mathbb{R}P^n, \mathbb{Z}) = \mathbb{Z} \text{ if } * = n \text{ is odd}$$

$$= \mathbb{Z}_2 \text{ if } * \text{ is odd and less than } n$$

$$= 0 \text{ otherwise:}$$

Eg $H_*(\mathbb{R}P^2, \mathbb{Z}) = \begin{matrix} \mathbb{Z}_2 & * = 1 \\ \mathbb{Z} & * = 0 \end{matrix}$

and is otherwise 0, while

$$H_*(\mathbb{R}P^3, \mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 3 \\ 0 & * = 2 \\ \mathbb{Z}_2 & * = 1 \\ \mathbb{Z} & * = 0 \end{cases}$$

(Thus $H_*(\mathbb{R}P^n, \mathbb{Q})$ is (algebraically) isomorphic to $H_*(S^n, \mathbb{Q})$ if n is odd, but not if n is even.

Ex. A complex polynomial

$$f(z) = a_d z^d + \cdots + a_0 \in \mathbb{C}[z], \quad a_n \neq 0$$

of degree d defines a map

$$S^2 = \mathbb{C}_+ = \mathbb{C} \cup \infty \longrightarrow \mathbb{C} \cup \infty = S^2$$

and hence a homomorphism

cf $\boxed{\text{Rotman Ch I p 17}}$

$$f_* : H_*(S^2, \mathbb{Z}) \left\{ \begin{tikzcd} \mathbb{Z} \arrow[r, "* = 2", "\text{mult by } d"'] & \mathbb{Z} \\ \mathbb{Z} \arrow[r, "* = 0", "\text{identity}"'] & \mathbb{Z} \end{tikzcd} \right\} = H_*(S^2, \mathbb{Z})$$

[On the other hand

$$H_*(\mathbb{R}P^n, \mathbb{Z}_2) \cong \mathbb{Z}_2, \quad 0 \leq * \leq n$$

$$H_*(\mathbb{C}P^n, \mathbb{Z}_2) \cong \mathbb{Z}_2 \quad 0 \leq * \text{ \underline{even}} \leq 2n$$

$$\cong 0 \quad \text{otherwise}$$

Example The Lefschetz fixed-point formula

It is classical [Rotman] that certain spaces X (eg closed Euclidean balls) have the property that any self-map $f : X \to X$ has at least one fixed point x_0 (such that $f(x_0) = x_0$): as you can see by inspection, stirring your cup of tea smoothly always leaves at least one tea molecule in place.

The 'categorification' of the Euler invariant provided by homology allows us to associate invariants to *maps*, not just spaces; for example a self-map f of X, as above, defines a (graded) endomorphism

$$f_* : H_*(X, \mathbb{Q}) \longrightarrow H_*(X, \mathbb{Q})$$

of its rational homology, which has a *Lefschetz* number

$$L(f) = \sum (-1)^k H_k(X, \mathbb{Q}) \in \mathbb{Z};$$

[This is *a priori* only a rational number, but since it comes from an endomorphism of the underlying integral homology $H_*(X, \mathbb{Z})$, it must be an integer. For example, the Lefschetz number of the identity map $X \to X$ equals the Euler characteristic of X (since the trace of the identity map of a (f.dim'l) vectorspace is just the dimension of that vector space.

The Lefschetz fixed point formula asserts, roughly, that the Euler characteristic

$$\chi(F) = L(f)$$

of the fixed point set $F = \{x \in X \mid f(x) = x\}$ equals the Lefschetz number of f. [There are various ways of making this more precise; for example if F is a discrete set, and X is a manifold, then $L(f)$ can be expressed as a sum of local terms,* one for each fixed point. In particular, if $L(f) \neq 0$ then f has a fixed point]

Suppose for example that $T \in M_n(\mathbb{C})$ with $\det T \neq 0$ is an invertible $n \times n$ complex matrix; then T maps $\mathbb{C}^n - 0$ to $\mathbb{C}^n - 0$, commuting
$\underline{z} \mapsto T\underline{z}$

* See §3.8, defn of a fixed pt

with the action of $\mathbb{C}^\times$ on $\mathbb{C}^n - 0$ (ie $uT(\underline{z}) = T(u\underline{z})$) it thus induces a map

$$T_{\mathbb{P}} : \mathbb{CP}^{n-1} \to \mathbb{CP}^{n-1},$$

and hence a homomorphism

$$T_{\mathbb{P}*} : H_*(\mathbb{CP}^{n-1}, \mathbb{Z}) \longrightarrow H_*(\mathbb{CP}^{n-1}, \mathbb{Z})$$

of homology groups; but these groups are either $\mathbb{Z}$ or 0, $T_{\mathbb{P}}$ is invertible, and the only invertible endomorphisms of $\mathbb{Z}$ are multiplication by ± 1. It turns out that

$$L(T_{\mathbb{P}*}) = n \quad (\operatorname{Tr} T_{\mathbb{P}k} = 1 \text{ if } k = 0, 2, \dots, 2(n \quad = 0 \text{ otherwise}$$

so we should expect $T_{\mathbb{P}}$ to have n fixed points.

But a fixed point of $T_{\mathbb{P}}$ is a line through the origin fixed by T, and is thus defined by a vector $\underline{z} \in \mathbb{C}^n - 0$ such that

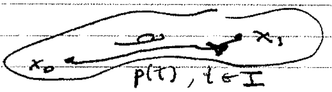
$$T(\underline{z}) = \lambda \underline{z} \quad , \lambda \in \mathbb{C}^\times ;$$

in other words, we expect T to have n nonzero eigenvectors.

(Note that a closed Euclidean ball has $H_*(B) = \mathbb{Z}$ $* = 0$, $= 0$ otherwise, and that any self-map $f : B \to B$ has $L(f) = 1 \dots$)

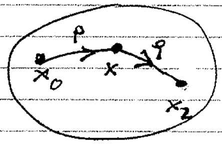
(3.4) § Paths and Homotopies

The set $\pi_0(X)$ of path components of a topological space was cited informally in Part I of these notes (p xx); it is considered more formally in Rotman (Ch I p26):



Defn A path, from x_0 to x_1, in a space X is a function $p \in \mathrm{Maps}(I, X)$ such that $p(0) = x_0$, $p(1) = x_1$ ($I = [0,1] \subset \mathbb{R}$).

Claim If p is a path from x_0 to x_1, and q is a path in X from x_1 to x_2, then there is a $\overset{\text{in } X}{}$ composite path $\overset{q * p}{}$ [Th 1.15, via the glueing lemma [Ch I p 14–15]:



This defines an equivalence relation on X ($x_0 \sim x_1 \Leftrightarrow \exists$ path from x_0 to x_1): it is transitive by the claim above, symmetric since $\bar{p}(t) := p(1-t)$ defines a path from x_1 to x_0 if p defines a path from x_0 to x_1, etc.

The set $\pi_0(X)$ is the quotient of X under this equivalence relation, and

[Rotman]

Thm 1.16: $X \mapsto \pi_0 X : (\mathrm{Top}) \longrightarrow (\mathrm{Sets})$ is a functor, from spaces to sets.

[For if $p : I \to X$ is a path in X, and $f : X \to Y$ is a map from X to Y, then $p \circ f : I \to X \to Y$ is a path in Y; so if $x_0 \sim x_1$ in X, then $f(x_0) \sim f(x_1)$ in Y, and we get a map

$$\pi_0 X = X/\sim \;\longrightarrow\; Y/\sim\; = \pi_0 Y$$

$\overbrace{\text{of equivalence classes.}}^{\text{sets of}}$]

Remark We will see later that $H_0(X, \mathbb{Z})$ is the free abelian group generated by $\pi_0 X$ (and hence that $H_0(X, \mathbb{Q})$ is a $\mathbb{Q}$-vector space of dimension $\#(\pi_0 X)$).

Ex If $X = \{1, \dots, n\}$ (with the discrete topology) then $\#(\pi_0 X) = n$ [duh!].

Note that in general

$$\mathrm{Map}(X, Y\times Z) \cong \mathrm{Map}(X,Y) \times \mathrm{Map}(X,Z):$$

a function $f: X \to Y\times Z$, $f(x) = (f_1(x), f_2(z))$, defines functions $f_1 : X \to Y$ and $f_2\ X \to Z$ such that

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[dr, "f"'] & X\times X \arrow[d, "f_1\times f_2"] \\ & Y\times Z \end{tikzcd}$$

, $\Delta(x) = (x,x)$

commutes. Thus a |

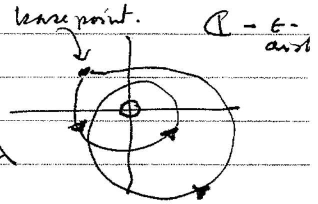
In particular, a path in $X\times Y$ defines 2 paths in X and Y, and conversely. The equivalence relation defined by paths on $X\times Y$ is thus the product relation of the path relations on of the component spaces, so

$$\pi_0(X\times Y) \cong \pi_0(X) \times \pi_0(Y),$$

[cf. Rotman Thm 3.7 p 46].

Although it is discussed in more detail later [Rotman ch 3 p 44-49], this seems like a good place to say a few words about the fundamental group $\pi_1(X,x)$ of a pointed space:

Defn Let X be a (nonempty!) space, with $x \in X$ its base point. A loop in X, based at x, is a continuous function $\lambda : I \to X$ such that $\lambda(0) = \lambda(1) = x$. Alternately, by the glueing lemma, we can think of λ as a map from the circle (eg $\mathbb{T} \in \mathbb{C}$ with $1 \in \mathbb{C}$ as basepoint, or $\mathbb{R}_+$ with ∞ as basepoint...) to X.



[Recall [Rotman Ex 0.2, 0.6, 0.7 p 7-8] that the category of pairs of topological spaces has as objects, *inclusions* $(X,A) := X \supset A$, $(Y,B) = (Y \supset B)$ of a subspace into a topological space. A map

$$f : (X,A) \to (Y,B)$$

of pairs is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f_{|A}"] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

(defined by a function from X to Y with the property that $f(A) \subset B$). The category of pairs of spaces contains the subcategory of *pointed spaces*, (X,x), i.e. pairs such that the subset A is a single point; thus a map $\lambda : (S^1, 1) \to (X,x)$ of pointed spaces is a loop, based at $x = \lambda(1)$.

Definition A *homotopy* $F : f_0 \simeq f_1 : X \to Y$ between $f_0, f_1 \in \mathrm{Map}(X,Y)$ is a *path* from f_0 to f_1.

Alternately: F is a continuous function

$$F : I \to \mathrm{Map}(X,Y),$$

such that $F(0) = f_0$, $F(1) = f_1$.

Alternately: Since

$$\mathrm{Maps}(I, \mathrm{Maps}(X,Y)) \cong \mathrm{Maps}(I \times X, Y)$$

for "reasonable" spaces (part I p xii), a homotopy from f_0 to f_1 is a function

$$F : I \times X \to Y$$

such that $F(0,x) = f_0(x)$, $F(1,x) = f_1(x)$.

Definition

$$\Omega(X,x) = \{\lambda \in \mathrm{Map}(S^1, X) \mid \lambda(1) = x\}$$

is the *space of loops*, based at x, in X. It is a *closed* subspace of the space of all maps from S^1 to X, in the compact open topology; at least if X is reasonable.

xxxi)

<u>Claim</u>, $(X, x) \longmapsto (\Omega(X,x), e_x)$, where $e_x : I \to x \in X$ is the <u>constant</u> map to X, is a functor, from the category of pointed spaces to itself.

<u>Remark</u> Spaces of maps in <u>general</u> define functors: for example a reasonable (eg compact) space Z defines a functor

$$X \longmapsto Maps(Z,X) \; : \; (Spaces) \to (Spaces),$$

for if $f : X \to Y$ is a map of spaces, then

$$Maps(Z,X) \ni h \mapsto f \circ h \in Maps(Z,Y)$$

is the induced map. If $W \xrightarrow{g} X \xrightarrow{f} Y$ is a composition, then the induced diagram

$$\begin{array}{ccccc} k & \longmapsto & k\circ g & \longmapsto & (k \circ g)\circ f \\ Maps(Z,W) & \longrightarrow & Maps(Z,X) & \longrightarrow & Maps(Z,Y) \end{array}$$

equals (by associativity) the map

$$\begin{array}{ccc} k & \mapsto & k \circ (g \circ f) \\ Maps(Z,W) & \longrightarrow & Maps(Z,Y) \end{array}$$

induced by $g \circ f$.

xxxii)

Similarly,

$$X \longmapsto Maps(X,Z)$$

defines a <u>contravariant</u> functor

$$\begin{array}{ccc} h & \longmapsto & f \circ h \\ Maps(Y,Z) & \longrightarrow & Maps(X,Z). \end{array}$$

We can do similar constructions for pairs, or for pointed spaces, by taking

$$Maps((X,A),(Y,B)) = \{ f \in Maps(X,Y) \mid f(A) \subset B$$

<u>Definition</u> The <u>fundamental</u> group of a space X, based at $x \in X$, is the set of components

$$\pi_0 \Omega(X,x) := \pi_1(X,x)$$

of the based loop space. I'll defer the proof that this is in fact [Rotman Th 3.3 p 44] a group till later; but note that $e_x \in \Omega(X,x)$ defines a base point in $\pi_0 \Omega(X,x)$ (which turns out to be the identity element in the group structure).

Since both Ω and π_0 are functors, their composition $\pi_1 := \pi_0 \circ \Omega$ is a functor.

Moreover, since $\mathrm{Maps}(X, Y\times Z) \cong \mathrm{Maps}(X,Y)\times \mathrm{Maps}(X,Z)$ and since π_0 preserves products, we have

$$\pi_1(X\times Y, x_0\times y_0) \cong \pi_1(X,x_0)\times \pi_1(Y,y_0).$$

3.5 Conventions involving products of pairs of spaces

If $f:(X,A)\longrightarrow(V,C)$ and

$g:(Y,B)\longrightarrow(W,D)$ are maps of pairs, then obviously $f\times g$ maps $X\times Y$ to $V\times W$, but it may not be so obvious (please check!) that it maps $X\times B\cup A\times Y$ to $V\times D\cup C\times W$.

This leads to the definition

Note $A=\emptyset \Rightarrow$
$X\times(Y,B) = (X\times Y, X\times B)$

$$(X,A)\times(Y,B) := (X\times Y,\ X\times B\cup A\times Y)$$

for the product in the category of pairs of spaces.

The product

$$(X,x)\times(Y,y) := (X\times Y,\ x\times Y\cup X\times y)$$

of a pair of pointed spaces is therefore *not* again a pointed space — unless we do something about it.

Definition The smash product

Rotman Ch11 p333

$$(X,x)\wedge(Y,y) := (X\times Y)/(x\times Y\cup X\times y)$$

of two pointed spaces is the pointed space obtained from $X\times Y$ by collapsing the (closed) subspace $x\times Y\cup X\times y$ to a point. There are natural associativity homeomorphisms

$$((X,x)\wedge(Y,y))\wedge(Z,z)\cong(X,x)\wedge((Y,y)\wedge(Z,z))$$

etc.

Note that the one-point compactification X_+ of a locally compact space X is pointed by point at infinity. [If X is already compact, we understand X_+ to be the union of X with a disjoint basepoint.]

Proposition If X and Y are locally compact spaces then $X_+ \wedge Y_+$ is homeomorphic to the one-point compactification of $X \times Y$.

Ex $\mathbb{R}^n_+ \cong S^n$ and $\mathbb{R}^m_+ \cong S^m$ so

$$S^n \wedge S^m \cong S^{n+m}$$

(with the convention/abuse of notation that allows us to write smash products without explicitly indicating the basepoint, eg when the spaces involved are path-connected).

possible convention: if X is a pointed space, $*_X \in X$ is its basept.

Remark If (X,A) is a pair of spaces with A closed, then the quotient space X/A obtained by collapsing A to a point is Hausdorff and

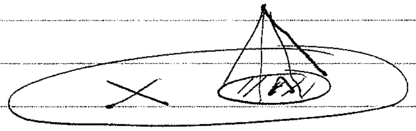
Ex the *wedge product* $X \vee Y$ of two pointed spaces $(X, *_X)$, $(Y, *_Y)$ is the quotient $(X \sqcup Y)/*_X \sim *_Y$; it satisfies $X \wedge (Y \vee Z) = X \wedge Y \vee X \wedge Z$, etc. **Ex** $\vee S^n$ = bouquet of spheres

there is a functor from the category of such pairs to pointed spaces, which interprets X/A as a space with basepoint $\{A\}$

A slightly better behaved construction associates to a pair (X,A) the pointed space $X \cup_A CA$, where the cone CA on A is the pointed space

$$CA = (A \times I)/(A \times 1),$$

[Rotman ch I p23], and $X \cup_A CA$ is the quotient of the disjoint union of X and CA which identifies $A \subset X$ with $A \times 0 \subset CA$ in the obvious way:



The map $X \cup_A CA \to (X \cup_A CA)/CA = X/A$ is a *homotopy equivalence* under reasonable hypotheses on A ...

what are they?!

Rotman 8.5 p187
Cor 8.10 p190
8.33 p212

<u>Definition</u> If (X,x) and (Y,y) are pointed spaces let $\mathrm{Map}_*(X,Y) := \mathrm{Map}((X,x),(Y,y))$ denote the space of pointed maps between them. It is itself a pointed space, with the constant map $Y \to y$ as basepoint.

<u>Proposition</u> If X, Y, Z are reasonable pointed spaces, then

$$\mathrm{Map}_*(X, \mathrm{Map}_*(Y,Z)) \cong \mathrm{Map}_*(X \wedge Y, Z).$$

<u>Corollary</u> If we define

$$\Omega^n(X) := \mathrm{Map}_*(S^n, X)$$

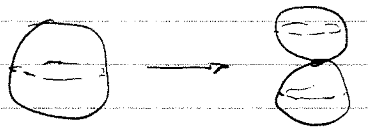
then $\Omega^m(\Omega^n(X)) = \Omega^{m+n}(X)$

[because $\Omega^m(\Omega^n(X)) = \mathrm{Map}_*(S^m, \mathrm{Map}_*(S^n, X))$

$$\cong \mathrm{Map}_*(S^m \wedge S^n, X) \cong \Omega^{m+n}(X)]$$

<u>Definition</u> $\pi_n(X,*) = \pi_0\,\Omega^n(X)$ is the n^{th} homotopy group of X.

[We won't need the group structure* till later, so I won't spell out the proof, but its based on the idea that the space S^n/S^{n-1} obtained from the sphere by collapsing its equator



is the wedge sum $S^n \vee S^n$

Rotman Ch 11
p332

(where $(X,x) \vee (Y,y)$ is the quotient of the disjoint union of X and Y which glues the basepoints x and y together). By the glueing lemma, two maps $f, g\ (S^n,*) \to (X,*)$ define an element $f \vee g \in \mathrm{Maps}_*((S^n \vee S^n, *), (X,*))$, and the collapse map $S^n \to S^n \vee S^n$ sends this to an element of $\mathrm{Maps}_*((S^n,*),(X,*))$ using the contravariant functoriality of $\mathrm{Maps}(-,X)$.]

---

* It follows from $\pi_n(X,*) = \pi_1(\Omega^{n-1}X, *)$.
See Rotman Ch 11.21 p 335 for the nice proof that π_1 of a group object (eg π_2) is abelian.

Remark by the argument on p ix),

$$\Omega^n(X\times Y, x\times y) \cong \Omega^n(X,x)\times\Omega^n(Y,y)$$

so $\pi_n(X\times Y, x\times y) \cong \pi_n(X,x)\times\pi_n(Y,y)$.

DEFINITION The homotopy category (HoT) (symmetric monoidal, ie it has a product $X,Y \to X\wedge Y$) (of reasonable pointed spaces, cf [Rotman p 16] has reasonable pointed spaces as its objects, with

$$\mathrm{Maps}_{\mathrm{HoT}}(X,Y) := \pi_0\,\mathrm{Maps}_*(X,Y).$$

Since π_0 respects products, the associativity maps

$$\mathrm{Maps}_*(X,Y)\times\mathrm{Maps}_*(Y,Z)\to\mathrm{Maps}_*(X,Z)$$

imply the existence of good associativity maps in (HoT).

[In other words, maps in the homotopy category are equivalence classes, with respect to homotopy, of continuous maps of pointed topological spaces.]

③.⑥ § Back to the Axioms!

The Eilenberg-Steenrod axioms for a homology theory were introduced informally in Part I (p xix). These notes — or, at least, this introduction to the study of algebraic topology — argues for a rather strict analogy (ie 'categorification') between classical measure theory and homology theory, ie

$X \longmapsto \mu(X)$, a function

Measure Theory: Metric spaces $\longrightarrow$ the set real numbers

$\Updownarrow$

Homology Theory: (Topological Spaces) $\longrightarrow$ the category of (graded) Abelian grps

$X \longmapsto H_*(X)$, a functor

Under this analogy the characteristic property

$$\mu(U\cup V)+\mu(U\cap V) = \mu(U)+\mu(V)$$

of a finite measure corresponds to the

Mayer-Vietoris *exact* *triangle*

$$\begin{tikzcd}
H_*(U\cap V) \arrow[rr] & & H_*(U)\oplus H_*(V) \arrow[dl] \\
& H_*(U\cup V) \arrow[ul, "(-1)"] &
\end{tikzcd}$$

of graded abelian groups, while

Fubini's *theorem* $\mu(U\times V) = \mu(U)\cdot\mu(V)$

Rotman Ch 9 p 265

correspond with the *Künneth* *formula*

$$H_*(U\times V, F) \cong H_*(U, F)\otimes_F H_*(V,F)$$

for homology with field coefficients.

Note, however, that there are important differences: a measure is a *function*, from sets to real numbers, while a homology theory is a (taking values in some category, eg vector spaces) *functor*; which, to be precise, is defined not on the category of spaces itself, but on its quotient homotopy category (HoT).

This subsection is concerned with some consequences of these axioms, and the statement of some of their technical variants: in particular, the *excision* axiom.

Apology: I should have said earlier that, for any commutative ring R (eg $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}_n$) there is a version $H_*(X, R)$ of homology with coefficients in the category of R-modules; but I have been trying to keep things simple. I will continue by taking $R = \mathbb{Z}$, ie R-Mod = Abelian groups.

Definition Any space X has a unique *collapse* map $X \to \mathrm{pt}$. Define its *reduced* homology to be

$$\tilde{H}_*(X,\mathbb{Z}) := \text{Kernel}\,[\,H_*(X,\mathbb{Z}) \to H_*(\mathrm{pt},\mathbb{Z})\,]$$

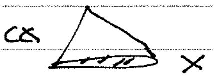
Remark It follows from the Meyer Vietoris axiom that if $X = \amalg X$ is the disjoint union of a finite number of path components X_α, that

$$\boxed{H_*(X) \cong \bigoplus_\alpha H_*(X_\alpha);}$$

and since each X_α is path-connected, we have $H_0(X_\alpha, \mathbb{Z}) \cong \mathbb{Z}$. In particular, if X itself is path-connected,

$$\tilde{H}_0(X, \mathbb{Z}) = 0.$$

This simplifies many arguments induction on dimension.



Definition The cone $CX = (X \times [0,1]) / ((x,1) \sim *) \supset X$ is a contractible space [Rotman Ch I p 23], homotopy equivalent to a point; thus $H_*(CX, \mathbb{Z}) = \mathbb{Z}$, $* = 0$

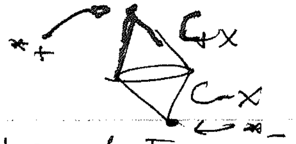
$= 0$ otherwise

The suspension $\Sigma X = C_+X \cup_X C_-X$ of X [Rotman Ch 11 p 329] is the union of two cones, $C_+X = (X \times [0,1]) / ((x,+1) \sim *_+)$ and

$$C_-X = (X \times [-1,0]) / ((x,-1) \sim *_-),$$

glued together along $C_+X \cap C_-X \cong X \times 0$.

$$\tilde{H}_*(\vee X_\alpha) = \bigoplus \tilde{H}_*(X_\alpha)$$



The reduced suspension of a pointed space X is the quotient $\Sigma X / (x \times [-1,+1])$; it is therefore a pointed space. In fact

$$\Sigma : \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$$

is a functor, from the category of pointed spaces to itself, and

$$\Sigma(X) \cong S^1 \wedge X$$

(with both S^1 and X interpreted as pointed spaces: for $S^1 \wedge X \cong ([-1,+1] \times X) / (\{\pm 1\} \times X \cup [-1,+1] \times$

It is an easy corollary of the Mayer-Vietoris exact triangle axiom, that for spaces U, V (eg with finitely many components) there is an exact triangle

$$\begin{tikzcd} \tilde{H}_*(U\cap V) \arrow[rr] & & \tilde{H}_*(U)\oplus\tilde{H}_*(V) \arrow[dl] \\ & \tilde{H}_*(U\cup V). \arrow[ul, "(-1)"] & \end{tikzcd}$$

Proposition $\tilde{H}_*(\Sigma X) \cong \tilde{H}_{*-1}(X)$ (for X pointed, and $\Sigma X \cong S^1 \wedge X$).

Proof Take $U = C_+X$ and $V = C_-X$: both are contractible, hence $\tilde{H}_*(C_\pm X) = 0$. But then $\tilde{H}_*(U\cup V) \cong \tilde{H}_*(\Sigma X)$ is isomorphic to $\tilde{H}_{*-1}(U\cap V) = \tilde{H}_{*-1}(X)$.

Remark This is intended to shed some

light on the role of the grading of $H_*(X)$: For example, $\tilde{H}_*(S^0;\mathbb{Z}) \cong \mathbb{Z}$ (since S^0 has *two* path-components); so

$$\tilde{H}_*(\Sigma^n S^0, \mathbb{Z}) \cong \tilde{H}_*(S^n, \mathbb{Z}) \cong \tilde{H}_*(S^0),$$

ie $\tilde{H}_*(S^n) \cong \mathbb{Z}$ if $* = n$, $= 0$ otherwise.

3.7 —————— " ——————

A variant of the finite-measure identity for the Euler characteristic asserts that if A is a reasonable closed subspace of X (so the quotient space X/A is Hausdorff), then

$$\chi(X) = \chi(A) + \chi(X/A).$$

Taking $U = X$, $V = CA$ in the (reduced) Mayer-Vietoris triangle (so $U \cap V = A$)

gives

$$\begin{tikzcd} \tilde{H}_*(A) \arrow[r] & \tilde{H}_*(X) \oplus \overset{0}{\overset{\shortparallel}{\tilde{H}_*(CA)}} \arrow[d] \\ & \tilde{H}_*(X \cup CA) \arrow[ul, "\deg(-1)"'] \end{tikzcd} ;$$

a more standard notation for the group at the bottom is

$$H_*(X,A) := \tilde{H}_*(X \cup CA),$$

in which we get the "long exact sequence

$$\cdots \to H_*(A) \to H_*(X) \to H_*(X,A) \to H_{*-1}(A) \to$$

for *relative* homology'. It is helpful to think of this as coming from the sequence

$$A \to X \to X \cup CA \to (X \cup CA)/A = \Sigma A \to \Sigma X$$

of maps of (pointed) spaces, via the isomorphism

$$\tilde{H}_*(\Sigma A) \cong \tilde{H}_{*-1}(A).$$

Rotman Ch 11 p 350

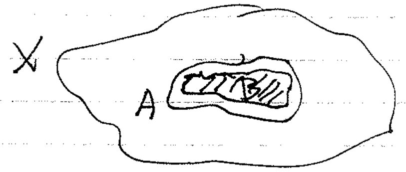
The usual presentation of the homology axioms takes the homology group $H_*(X,A)$ of pairs of spaces as a primitive object and defines

$$H_*(X) = H_*(X,\emptyset).$$

This leads to the need for another (EXCISION) axiom.

Recall that the *closure* $\overline{B}$ of a subset $B \subset X$ is the smallest closed subset $\overline{B} \supset B$ of X ($\overline{B}$ is the intersection of all the closed subsets of B containing X), and that the *interior* $A^\circ \subset A$ of a subset A of X is the union of all the open subsets of A (ie $A^\circ \subset A$ is the largest open in the topology of X subset of A).

The excision axiom asserts that if $B \subset A \subset X$ are such that $\overrightarrow{B} \subset A^0$



ie that $\overline{B} \cap (A - A^0) = \phi$,
B does not get too
close to the "boundary"
$A - A^0 \subset A$

Then the map

$$(X - B, A - B) \longrightarrow (X, A)$$

of pairs induces an isomorphism

$$H_*(X-B, A-B) \xrightarrow{\cong} H_*(X, A)$$

on homology.

Proof (using the definition of relative homology above): this follows because the inclusion

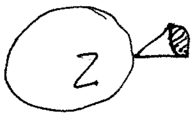
$$(X-B) \cup C(A-B) \to X \cup CA$$

$$= ((X-B) \cup C(A-B)) \cup CB$$

(note that $((X-B) \cup C(A-B)) \cap CB$ = the cone point)

is a homotopy equivalence. More generally,

Lemma: the inclusion



$$Z \longrightarrow Z \cup CB,$$

with $Z \cap CB = pt$, is a homotopy equivalence

Proof I claim that the map

$$Z \cup CB \to Z$$

which collapses CB to the cone point of Z is a homotopy inverse to the inclusion: since

$$1_Z : Z \to Z \cup CB \to Z$$

is the identity it suffices to show that the other composition

$$c : Z \cup CB \to Z \to Z \cup CB$$

is homotopic to the identity. but $c|_Z = 1_Z$ is the identity on Z, and CB is contractible; so we have a homotopy of the composition

$$CB \to pt \longrightarrow CB$$

to the identity map of CB.

Now the closure $C\bar{B}$ of the cone on B misses the cone on the boundary of A, so we can glue these homotopies together to define the promised homotopy of c with the identity map of $Z \cup CB$.

(3.8) § Some applications

Proposition 3.8.1

$$H_*(\mathbb{R}^n, \mathbb{R}^n - \{0\}, \mathbb{Z}) \cong \tilde{H}_{*-1}(S^{n-1}, \mathbb{Z}) \cong \mathbb{Z} \text{ if } * = n$$
$$= 0 \text{ otherwise}$$

Proof: Since $\mathbb{R}^n$ is contractible, the exact sequence

$$\cdots \to \underset{\substack{\| \\ 0}}{H_*(\mathbb{R}^n)} \to H_*(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \to H_{*-1}(\mathbb{R}^n - 0) \to \underset{\substack{\| \\ 0}}{H_{*-1}(\mathbb{R}^n)} \to$$

implies that

$$H_*(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \xrightarrow{\cong} H_{*-1}(\mathbb{R}^n - 0),$$

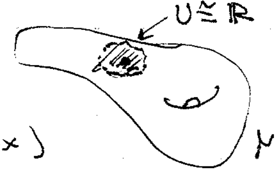
but $\mathbb{R}^n - 0 \cong S^{n-1} \times \mathbb{R}_+$ (polar coordinates), so

$$H_{*-1}(\mathbb{R}^n - 0) \cong H_{*-1}(S^{n-1})$$ by homotopy eq.

Corollary If M is an n-dimensional manifold, and $x \in M$, then

$$H_*(M, M-x, \mathbb{Z}) \cong \mathbb{Z} \quad * = n,$$
$$= 0 \quad \text{otherwise}$$

Proof Since M is a manifold, x has a neighborhood $U \subset M$ homeomorphic to an open set in Euclidean space, which we may take to be an open ball in $\mathbb{R}^n$. But now



$$H_*(U, U-x) \xrightarrow{\cong} H_*(M, M-x)$$

is an isomorphism by the excision axiom, since

$$(U, U-x) = (M-(M-U), (M-x)-(M-U)).$$

[Note that since U is open, M-U is closed, so

$$M-U = (M-U)^c \subset (M-x)^{int} = M-x$$

(because $\{x\}$ is closed, hence M-x is open).

Remark Peano's construction (in the early 20th century) of a continuous map from the unit interval *onto* the unit square raised the question of *invariance of dimension*: could $\mathbb{R}^n$ be homeomorphic to $\mathbb{R}^m$, for $n \neq m$? The argument above says that this *can't happen*.

[Related applications (eg the Jordan curve theorem are discussed in Rotman Ch 6 (127 – 130) or in Hatcher

Definition The *orientation bundle*

$$M^{or} := \coprod_{x \in M} H_n(M, M-\{x\}, \mathbb{Z}) \xrightarrow{\pi} M$$

can be topologized to have a cover $\{\tilde{U}_i\}$ by open sets, such that each $\tilde{U}_i \cong \mathbb{Z} \times U_i$ for $\{U_i\}$ an open cover of M.

More precisely, M has a covering $\{U_i\}$ by open sets homeomorphic to (open) Euclidean n-balls, and for any such U we have homeomorphisms

$$\coprod_{x \in U} H_n(M, M-x) \cong \coprod_{x \in U} H_n(U, U-x)$$

$$\cong \coprod_{x \in U} H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong \mathbb{Z} \times U,$$

defining *local trivializations* of the orientation bundle.

M^{or} is thus a bundle of discrete abelian groups (with fiber $\cong \mathbb{Z}$). The theory of *covering spaces* [Rotman Ch10. 41, p300] implies that this covering is classified by a group homomorphism $\pi_1(M) \to \mathrm{Aut}_{(Ab)}(\mathbb{Z}) = \{\pm 1\} \cong \mathbb{Z}_2$. M is orientable if this homomorphism is *trivial*, ie if $M^{or} \cong \mathbb{Z} \times M$; otherwise M is *not* orientable!

The orientation bundle is central to the proof of Poincaré's duality theorem, which perhaps justifies the following digression (to which we will return in §8).

<u>Definition</u> The open subsets U of a topological space X form a category $\mathrm{Open}(X)$, with inclusions as the maps.

A <u>contravariant</u> functor

$$F : \mathrm{Open}(X)^{op} \longrightarrow Ab$$

(ie $U_0 \subset U_1 \Rightarrow \exists$ homomorphism "restriction"

$$r^{U_0}_{U_1} : F(U_1) \longrightarrow F(U_0)$$

a similar definition can be made for <u>set</u>-valued functors

such that $\forall\, U_0, U_1 \in \mathrm{Open}(X)$, the sequence

$$0 \to F(U_0 \cup U_1) \xrightarrow{r^0_{01} \oplus r^1_{01}} F(U_0) \oplus F(U_1) \xrightarrow{r^{01}_0 - r^{01}_1} F(U_0 \cap U_1)$$

is <u>exact</u>. <u>Example</u>

$C(U)$ = continuous fns $U \to \mathbb{R}$ or $\mathbb{C}$
$C^\infty(U)$ = <u>smooth</u> fns, $U \subset$ manifold

(More generally $F(A)$ can be defined for any $A \subset X$, if $\underline{F}$ is a sheaf.

<u>Ex</u> <u>holomorphic</u> or <u>algebraic</u> functions, of a complex manifold or an algebraic variety.

In fact it can be shown that for any sheaf F on a (Hausdorff) topological space X, there is a (possibly non-Hausdorff) space $\underline{F} \xrightarrow{\pi_F} X$ such that its set of <u>sections</u> is

$$F(U) = \{ s : U \to \underline{F} \mid \pi_F \circ s = 1_U \} .$$

"Sections" of F, ie elements of $F(U)$, can thus be <u>patched</u> <u>together</u> like classical functions.

<u>Note</u> that sections of a vector bundle $V \to X$ define a sheaf, but that the sheaf space $\underline{V} \to X$ is very different from V!

<u>Proposition</u> If M is an n-dimensional manifold,

$$U \mapsto H_n(M, M-U) := M^{or}(U)$$

is a sheaf.

Lvii)

<u>Proof</u>. The required exactness property is a consequence of the Mayer-Vietoris exact sequence, together with a little fiddling with excision:

I claim that $M/M-U \cong U_+$, and that the fiber product

$$\begin{tikzcd}
 & Z \arrow[dl, dashed] \arrow[dr, dashed] & \cong M/(M-U_0\cap U_1) \cong U_{0+}\cup U_{1+} \\
M/M-U_0 \arrow[dr] & & M/M-U_1 \arrow[dl] \\
 & M/(M-(U_0\cup U_1)) \cong U_{0+}\cap U_{1+} &
\end{tikzcd}$$

(ch I p iii)

yields a long exact sequence

$$\cdots \to \tilde{H}_*((U_0\cup U_1)_+) \to \tilde{H}_*(U_{0+}) \oplus \tilde{H}_*(U_{1+}) \to \tilde{H}_*(U_{0+}\wedge U_{1+}) \to$$

The fundamental "geometric" fact behind the Poincaré duality theorem is the following

L~~x~~iii)

$\boxed{\text{Moreover, } H_k(M,M-A) = 0 \text{ for } k>n}$

<u>Proposition</u> $\forall\, A$ closed $\subset \overset{n\text{-dim'l}}{M}$ $\exists$ natural iso

$$H_n(M, M-A) \longrightarrow M^{\underline{or}}_c(A) = \left\{ s : A \to \underline{\underline{M}}^{or} \;\middle|\; \begin{array}{l} 1)\ \pi\circ s = 1_A \text{ and} \\ 2)\ \text{the closure of the set of } x\in A \text{ such that } s(x)\neq 0 \text{ is } \underline{\text{compact}} \end{array} \right\}$$

<u>Corollary</u>, $H_n(M) = M^{\underline{\underline{or}}}_c(M)$ is the set of global sections of $\underline{\underline{M}}^{or}$ with compact support.

Since sections of $\underline{\underline{M}}^{or}$ are <u>locally constant</u>, this implies $H_n(M) = 0$ if M is <u>not</u> compact, or not orientable; and that, if M <u>is</u> compact orientable, then eg

$$H_n(M, \mathbb{Z}) \cong \mathbb{Z}.$$

<u>Remark</u> The bundle $\Lambda^n(T^*M)$ of n-forms on a manifold is of real fiber dimension one; it

may, or may not, be isomorphic to the trivial bundle $M \times \mathbb{R}$. The n^{th} de Rham cohomology group $H^n_{dR}(M,\mathbb{R}) \cong \mathrm{Hom}(H_n(M,\mathbb{Z}),\mathbb{R})$ is one-dimensional in the latter case, and it can be interpreted as generated by the *volume form*

$$d\,vol_g := |\det g_{ik}|^{1/2} dx_1 \wedge \cdots \wedge dx_n \in \Omega^n(M)$$

defined by a Riemannian metric g.

(i.e. a generator of H_n has been chosen

3.8.2

Example If M, M' are two compact *oriented* manifolds of the same dimension $n = \dim M = \dim M'$ and $f : M \to M'$ is a continuous map, then

$$f_* : H_n(M,\mathbb{Z}) \cong \mathbb{Z} \longrightarrow H_n(M',\mathbb{Z}) \cong \mathbb{Z}$$

determines an element $\deg(f) \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$, called the *degree* of f.

For example

$$z \mapsto z^n : \mathbb{C} \to \mathbb{C}$$

(or, more generally, any polynomial of degree n) defines an extension $S^2 \cong \mathbb{C}_+ \longrightarrow \mathbb{C}_+ \cong S^2$ represented on $H_2(S^2,\mathbb{Z}) \cong \mathbb{Z}$ as multiplication by $n = \deg(f)$.

If f is *smooth* then (by transversality) $f^{-1}(x)$ is generically a 0-dimensional manifold, i.e. a finite set of points, and it can be shown that, *generically*, $\# f^{-1}(x) = \deg f$.

Example Consider the *twist* map

$$T_{n,m} : S^n \wedge S^m \longrightarrow S^m \wedge S^n,$$
$$= \mathbb{R}^n_+ \wedge \mathbb{R}^m_+ = \mathbb{R}^{n+m}_+ \qquad = \mathbb{R}^m_+ \wedge \mathbb{R}^n_+ = \mathbb{R}^{m}$$

$T_{n,m}(x,y) = (y\,x)$ [illegible],

represented by the linear map $\begin{bmatrix} 0 & \mathbb{1}_m \\ \mathbb{1}_n & 0 \end{bmatrix} := \underline{\underline{T}}$

of $\mathbb{R}^{n+m}$ to itself. It is an easy calculation that $\det \underline{\underline{\tau}} = (-1)^{nm}$: in terms of exterior algebra we have $\det \underline{\underline{\tau}} = \Lambda^n(\underline{\underline{\tau}})$,

$$\Lambda^n(\underline{\underline{\tau}})(e_1\wedge\dots\wedge e_n\wedge f_1\wedge\dots\wedge f_m) = (f_1\wedge\dots\wedge f_m\wedge e_1\wedge\dots\wedge e_n),$$

with e_i (resp f_k) bases for $\mathbb{R}^n$ (resp $\mathbb{R}^m$).

But moving e_1 to the left gives $(-1)^m(e_1\wedge f_1\wedge\dots\wedge f_m\wedge e_2\wedge\dots\wedge e_m)$, and repeating this n times gives $((-1)^m)^n(e_1\wedge\dots\wedge e_n\wedge f_1\wedge\dots\wedge f_m)$. Arguing from de Rham cohomology, as above, implies that $\deg(\underline{\underline{\tau}}) = (-1)^{nm}$.

Example The *index* of an isolated fixed point

Suppose $f:(U,x_0)\to(U,x_0)$ maps an

open ball in $\mathbb{R}^n$ to itself, with x_0 an isolated fixed point: then

$$\tilde f(x) = \frac{x - f(x_0)}{|x - f(x_0)|}$$

maps some $B - x_0$ to S^{n-1}, where $B \subset U$ is again an open ball. But $B - x_0 \cong S^{n-1}\times\mathbb{R}$, so $\tilde f_{n-1} : H_{n-1}(B - x_0, \mathbb{Z}) \longrightarrow H_{n-1}(S^{n-1}; \mathbb{Z})$ is multiplication by an integer $\deg(\tilde f)$, called the *fixed-point index* of x_0 with respect to f.

Ex $(\mathbb{C},0)\ni z \xmapsto{f} z^n \in (\mathbb{C},0) \Rightarrow$

$$\tilde f(z) = \frac{z^n - 0}{|z^n - 0|} = e^{in\theta} \quad \text{if } z = re^{i\theta},$$

ie $\tilde f : S^1 \to S^1$ has degree n.

<u>Example</u> The cycle defined by a submanifold:

A compact oriented n-dimensional submanifold

$M \subset X$

of a space X defines a homomorphism

$H_n(M, \mathbb{Z}) \to H_n(X, \mathbb{Z})$ [illegible] ;

the image of a generator $[M] \in H_n(M, \mathbb{Z}) \cong \mathbb{Z}$ is the homology class in $H_n(X, \mathbb{Z})$ supported by M. For example, the class supported by the equator $S^1 \subset S^2$ of the two-sphere is <u>zero</u>. On the other hand, the class supported by the subspace $\mathbb{C}P_k$ of $\mathbb{C}P_n$, $n \geq k$, is a generator of $H_{2k}(\mathbb{C}P^n, \mathbb{Z})$.

Attaching a cell :

A continuous 'attaching' map $\alpha : S^n \to X$ from a sphere to a space X leads (cf p xxiii) else) to a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{H}_{*+1}(X') \arrow[r, "\partial"] & \tilde{H}_*(S^n) \arrow[rr, "\alpha_n"] \arrow[dr] & & \tilde{H}_*(X) \arrow[dddlll, bend left] \\
& & & \text{image } \alpha_n \arrow[ur] \arrow[dr] & \\
& & 0 \arrow[ur] & & 0 \\
& \tilde{H}_*(X') \arrow[r] & \tilde{H}_{*-1}(S^n) \arrow[r] & \cdots &
\end{tikzcd}$$

where $X' = X \cup_\alpha \text{Cone on } S^n \cong X \cup_\alpha B^{n+1}$ is said to be obtained from X by 'attaching an $(n+1)$-cell along the map α'.

Since $\tilde{H}_*(S^n)$ is concentrated in degree $* = n$, this implies the existence of a pair

Rotman 8.12 p154

$$0 \to \tilde{H}_{n+1}(X) \to \tilde{H}_{n+1}(X') \to \ker \alpha_n \to 0$$

$$0 \to \text{image } \alpha_n \to \tilde{H}_n(X) \to \tilde{H}_n(X') \to 0$$

(and that $\tilde{H}_*(X) \xrightarrow{\cong} \tilde{H}_*(X')$ if $* \neq n, n+1$).

Defn A cell complex structure on a space X is, roughly, iff $\exists$ a family

$$X_0 \subset X_1 \subset \cdots \subset X_k \ (:= sk_k X) \subset \cdots \subset$$

of subspaces, together with presentations

$$X_{n+1} = X_n \cup_{\amalg \alpha_i} (\amalg B^{n+1}_{(i)})$$

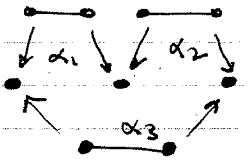
of X_{n+1}, as constructed inductively by attaching a collection of $(n+1)$-cells along attaching maps

$$\alpha_i : S^n_{(i)} \to X_n .$$

Ex A geometric simplicial complex (ie a polyhedr is such a cell complex, with its faces as the cells :

Ex ∂  $= \partial \Delta^2 =$  has

X_0 = 3 points, with 3 one-cells attached



$= X_0$

These exact sequences, in principle, give a method for computing the homology of a cell complex inductively; the problem, of course, is to understand the homology homomorphisms defined by the attaching maps. We will return to this eventually.

Ex The inclusion

$$\mathbb{C}^{n+1} \ni \underline{\underline{z}} \mapsto (z_0, \dots, z_n, 0) \in \mathbb{C}^{n+2}$$

defines a 1-to-1 map

$$\mathbb{CP}^n \ni [z_0 : \dots : z_n] \mapsto [z_0 : \dots : z_n : 0] \in \mathbb{CP}^{n+1}$$

Ex $\mathbb{CP}^0 = pt \hookrightarrow \mathbb{CP}^1 = \mathbb{C} \cup \infty$

On the other hand $\mathbb{C}^{n+1} \ni (z_0, \dots, z_n) \mapsto [z_0 : \dots : z_n : 1] \in \mathbb{CP}^{n+1}$

is also one-to-one

[$(z_0, \dots, z_n, 1) \sim (z_0', \dots, z_n', 1) \iff z_i' = u z_i$ $(\forall i)$, for some $u \in \mathbb{C}^\times$;

but $u = 1$ as well ...]

with $\mathbb{CP}^{n+1} - \mathbb{CP}^n$ as its image, ie set-theoretically

$$\mathbb{CP}^{n+1} = \mathbb{C}^{n+1} \sqcup \mathbb{CP}^n$$

$\mathbb{C}^{n+1} \leftarrow B^{2n}_0$ = interior of B^{2n+2}

is obtained by attaching a $2(n+1)$-cell to $\mathbb{CP}^n$ along a map $S^{2n+1} \longrightarrow \mathbb{CP}^n$, which turns out to be the quotient map

$$S^{2n+1} \longrightarrow S^{2n+1}/\mathbb{T} = \mathbb{CP}^n.$$

It follows now by induction on the sequences

$$0 \to \tilde{H}_{2n+2} \mathbb{CP}_n \longrightarrow \tilde{H}_{2n+2} \mathbb{CP}_{n+1} \longrightarrow \ker \partial_{2n+1} \to 0$$

$$0 \to \text{image } \partial_{2n+1} \longrightarrow \tilde{H}_{2n+1} \mathbb{CP}_n \longrightarrow \tilde{H}_{2n+1} \mathbb{CP}_{n+1} \to 0$$

that $\tilde{H}_* \mathbb{CP}_n = 0$ for $*$ odd
$= \mathbb{Z}$ for $* = 2k$, $1 \le k \le n$.

Part IV

1) Simplicial Complexes (4.1)

A corollary to the analogy between homotopy and measure theory is that the construction of both 1) requires a lot of new ideas, and 2) can be technically tedious.

This section is a sketch, focussing on an interesting special case, of the construction of a homotopy functor on a combinatorially accessible category of topological spaces. Later we'll reconsider these constructions at a higher level of resolution.

Ex Big data sets

Let X be a finite (but $\#(X) >> 0$) subset of a metric space $\mathcal{M}$, eg $\mathbb{R}^N$ for $N>>0$, and let $t \geq 0$ be a real parameter. It is useful to assume that $X = \{x_1, x_2, \ldots\}$ has been ordered.

ii)

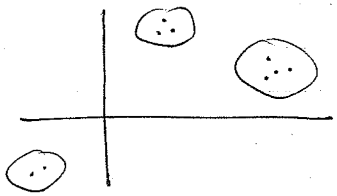
Define the t-thickening

$$X_*(t) = \{ X_n(t) \mid n \in \mathbb{N} \}$$

(classically, the (Vietoris)-Rips complex of X) in the family

$$X_n(t) := \{ S \subset X \mid \#S = n+1, \ x_i, x_k \in S \Rightarrow \operatorname{dist}(x_i, x_k) \leq t \}$$

of $(n+1)$-tuples of elements of X, all lying within a distance t of each other:



Observe that

1) $X_0(0) = X$

⊛ 2) If $S \in X_n(t)$ and $T \subset S$ with $\#T = m+1$, $m < n$, then $T \in X_m(t)$

3) if $t \leq t'$ then $\exists$ $(\forall n)$

$$i^t_{t'}(n) : X_n(t) \subset X_n(t')$$ such that

if $t \leq t' \leq t''$ then $i^{t'}_{t''} \circ i^t_{t'} = i^t_{t''}$

iii]

4) If $\phi : X \subset \tilde{X}$ takes finite, but not necessarily compatibly ordered, then $\exists \; X_n(t) \subset \tilde{X}_n(t) \ldots$

Property 2 asserts that the collection $X_*(t)$ is a (abstract!) simplicial complex:

Def'n [Rotman Ch 7 p 141] An (finite, ordered) simplicial complex K is a set of (nonempty) subsets of its totally ordered set $V(K)$ of vertices, satisfying

$$S \in K, \quad T \subset S \Rightarrow T \in K$$

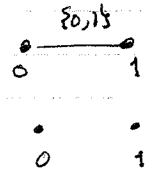
[Beware confusion between $\in$ and $\subset$!].

Elements of $K_n := \{ S \in K \mid \# S = n+1 \}$ are called n-simplices of K; thus if $S \in K_n$ and $T \subset S$ with $\# T = m+1$ then $T \in K_m$ is an m-simplex of K, said to be an m-dim'l (or (n-m)-codimensional) face of S.

iv)

Ex The standard n-simplex Δ^n is the set of nonempty subsets of $[n] = \{0, 1, \ldots, n\}$. It is the analog, in the soon-to-be defined category of simplicial complexes, of the closed Euclidean n-ball. The set of nonempty proper subsets of $[n]$ (ie distinct from $[n]$ itself), is also a simplicial complex, sometimes denoted $\partial \Delta^n$: it is an analog of S^{n-1}.

Ex Δ^1



$\sim B^1$

$\partial \Delta^1$ $\sim S^1$

Notation Since the vertices of K are ordered, a simplex S of K_n can be unambiguously denoted

$$S = \{ v_0 < v_1 < \ldots < v_n \} \text{ (or } [v_0, \ldots, v_n] \text{)}.$$

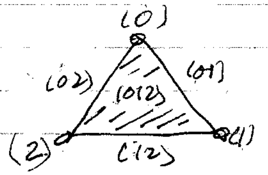
The i^{th} codimension-one face

$$S_{(i)} = [v_0, \ldots, \hat{v}_i, \ldots, v_n] \subset S, \quad 0 \leq i \leq n$$

v)

(opposite to the i^{th} vertex v_i of S) is obtained by deleting v_i from S [Rotman Ch 2 p 37].

<u>Ex</u>



($[2] \sim \Delta^2$ and its faces)

<u>Definition</u> A map $f : K \longrightarrow K'$ of simplicial complexes is a function $f : V(K) \longrightarrow V(K')$ on their vertex sets (<u>not</u> necessarily preserving their orders), such that if S is a simplex of K, then $f(S) = \{ f(s) \in K' \mid s \in S \}$ is a simplex of K'.

<u>Ex</u> The inclusions $\overset{\partial_i :}{\Delta^n_{(i)}} \subset \Delta^n$ $0 \leq i \leq n$ are maps of simplicial complexes. [<u>Rotman</u> calls them Σ! Ch 4 p64]

<u>Ex</u> The inclusion $\partial\Delta^n \subset \Delta^n$ is a map of simplicial cxes (analogous to $S^{n-1} \subset B^n$).

vi)

<u>Ex</u> $s_j : \Delta^n \longrightarrow \Delta^{n-1}$, defined by

[cf <u>Hatcher</u>,

$$s_j(i) = i, \quad i \leq j$$
$$= i-1, \quad i > j$$

(as a map from $[n] = \{0, \ldots, n\}$ to $[n-1] = \{0, \ldots, n-1\}$;

(eg :$[0123] \longrightarrow [012]$ sends both 2 & 3 to 2)

is a map of simplicial complexes.

<u>Ex</u> The maps $i_t^{t'} : X(t) \longrightarrow X(t')$ are maps of simplicial complexes.

<u>Definition</u> The m-skeleton $sk_m K$ of a simplicial complex K is the collection of k-simplices of K, with $k \leq m$. The inclusion $sk_m K \subset K$ is a map of simplicial complexes; moreover we have inclusions

$$\ldots \subset sk_{m-1} K \subset sk_m K \subset \ldots$$

which are maps of simplicial complexes.

<u>Exercise</u> $\partial_i \partial_j = \partial_{j-1} \partial_i$ if $i < j$ $\qquad$ $\partial_i s_j = s_j \partial_{i-1}$ if $i > j+1$

$\partial_i s_j = s_{j-1} \partial_i$ if $i < j$ $\qquad$ $s_i s_j = s_{j+1} s_i$ if $i \leq j$

$\partial_i s_j = id$ if $i = j$ or $j+1$

<u>Note</u> That any monotone (ie nondecreasing) map $f : S \longrightarrow T$ of finite ordered sets can be factored as a composition of s's and ∂'s.

vii)

K is said to be *n-dimensional* if n is the least integer such that $sk_n K = K$.

Definition [*Rotman* Ch 7 p143] If K is a simplicial complex, let

$$C_k(K) := \mathbb{Z}[\langle w_0,\dots,w_k\rangle|\dots]/(\text{Relations})$$

be the quotient of the free abelian group generated by symbols $\langle w_0,\dots,w_k\rangle$, where $\{w_i\}$ is a simplex of K, modulo the subgroup of relations generated by ($\sigma \in \Sigma_k$, $\text{sign}(\sigma) = \pm 1$)

$$\langle w_{\sigma(0)},\dots,w_{\sigma(k)}\rangle = \text{sign}(\sigma)\langle w_0,\dots,w_k\rangle;$$

NOTING THAT the w_i's may be repeated: in which case $\langle w_0,\dots,w_i\rangle = 0$ in $C_k(K)$!

Claim $C_k(K)$ is a free abelian group generated by symbols $\langle v_0,\dots,v_k\rangle$, where $v_0 < \dots < v_k$ is a simplex of K. In particular $C_k(K) = 0 \ \forall k > \overset{\dim K}{\overset{\shortparallel}{n}}$.

viii)

More generally, if M is a module over a commutative ring R, then

$$C_k(K,M) := C_k(K)\otimes_{\mathbb{Z}} M$$

(with M regarded as a $\mathbb{Z}$-module via $\mathbb{Z}\to R$).

Proposition If $f: K \to K'$ is a map of simplicial complexes, then

$$\forall k \quad f_*\langle w_i\rangle = \langle f(w_i)\rangle : C_k(K) \to C_k(K')$$

define a homomorphism of chain groups

$$f_* : C_*(K,M) \to C_*(K',M)$$

Remark *Note* that if f is *not* 1-1 on the simplex $S \in K_n$ (ie $f(S) \in K'_m$ with $m < n$) then $f_*(S) = 0$.

In other words, $K \mapsto C_*(K)$ is a functor from simplicial complexes to abelian groups (or, more generally, R-modules).

ix)

Ex. $i^{t'}_{t''*} \circ i^{t}_{t'*} = i^{t}_{t''*} : C_*(X(t)) \longrightarrow C_*(X(t''))$.

4.2

§ Geometric Realization

One advantage of simplicial complexes is that they are extremely rigid (in the sense that there are relatively few maps between them) and are ie their invariants are hence (relatively) effectively computable. A disadvantage is that they are not very economical: for ex a simplicial complex representing a 2-torus [Rotman Ch 7 p 133] requires at least 14 two-simplices. Another inconvenience is that there is no very easy definition of the product of two simplicial complexes.

→ simplicial sets work better; see 4.5 below

Definition If K is a simplicial complex, its geometric realization $|K| \subset \mathrm{Fns}(V(K), \overset{I}{[0,1]})$

x)

is the space of functions

$$\alpha : V \longrightarrow [0,1] = I$$

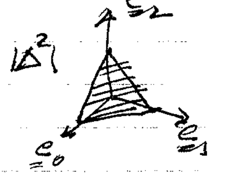
such that [Spanier Ch 3 §1 p 110]

$(\forall \alpha)$ $\{v \in K_0 \mid \alpha(v) \neq 0\} \in K$ (ie: is a simplex of K), and

$(\forall \alpha)$ $\sum_{v \in K_0} \alpha(v) = 1$

note that this is a metric space under the sup-norm because the vertex set is finite

Example Let $\underline{e}_i = (0, \ldots, 0, \overset{i^{th}\text{ place}}{1}, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ be the standard basis vectors. Then any vector $(\alpha_0, \ldots, \alpha_n) \in I^{n+1}$ such that $\sum \alpha_i = 1$ defines an element of $|\Delta^n|$; in particular $\{k \in \{0, \ldots, n\} \mid \alpha(k) \neq 0\}$ is a simplex of Δ^n.



Note that if S is a simplex, $|S|$ is convex: if $\alpha, \tilde{\alpha} \in |S|$, and $0 \leq t \leq 1$, then $t\alpha + (1-t)\tilde{\alpha} \in |S|$.

the geo realization is a geometric simplex, ie a convex set, which is the span of its vertic

⊛ **Proposition** : If $f: K \to K'$ is a map of simplicial complexes, $\alpha \in |K|$, then

$$(|f|(\alpha))(v') := \sum_{v \in K,\ f(v)=v'} \alpha(v) \in I$$

defines a continuous map

$$|f| : |K| \longrightarrow |K'|$$ of topological spaces

Proof $\boxed{\text{Exercise!}}$

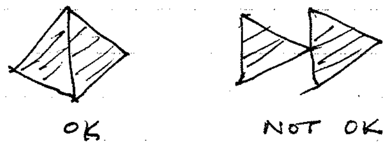
Note that if $S \in K$ is a simplex, then $|S|$ and $|f(S)|$ are geometric simplices, and that

$$|f| : |S| \longrightarrow |f(S)|$$

is a linear map: if $\alpha, \tilde{\alpha} \in |S|$, then

$$|f|(t\alpha + (1-t)\tilde{\alpha}) = t\,|f|(\alpha) + (1-t)\,|f|(\tilde{\alpha}).$$

Remark $|K|$ is a union of (geometric) simplices, such that any two intersect along a simplex (possibly empty) of positive codimension in either.



(# A map $f: K \to K'$ of simplicial cxes $\Rightarrow |f| : |K| \to |K'|$ of spaces filtered by their skeleta, $|sk_i K| \to |sk_i K'|$.
[$\exists$ category of filtered spaces & filtration-preserving maps.]

$|sk_n K|$ is a union of geometric simplices of dimension $\leq n$, and $|sk_{n-1} K|$ is a closed subspace. The quotient space

$$* \quad |sk_n K| / |sk_{n-1} K| \cong \vee S^n$$

is a bouquet of n-spheres: ex

$$|\Delta^n| / |\partial\Delta^n| \cong S^n \cong B^n/(\partial B^n \cong S^{n-1}).$$

Geo realization of the Rips cx

Example $|i_t^{t'}| : |X(t)| \subset |X(t')|$ is a continuous map, as is $|\phi| : |X(t)| \subset |\tilde{X}(t)|$.

Note that there are no natural maps relating the spaces $|X(t)|$ to the finite set X or the ambient space $M \supset X$.

p(i)

Rotman Ch 6 p113; Ex 7.11 p141

§ Barycentric subdivision $\boxed{\S 4.3}$

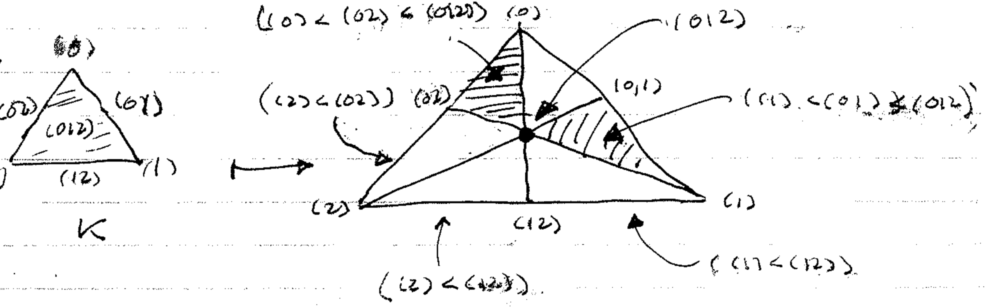
Definition The barycentric subdivision sd K of a simplicial complex K has vertex set $V(\text{sd}\,K) = K$;

a k-simplex of sd K is a chain

$$s_0 \subsetneq s_1 \subsetneq \dots \subsetneq s_k$$

of proper inclusions of simplices of K:

Ex



Ex. $\text{sd}[n] = \bigcup_{0 \le i \le n} \text{sd}_{(i)}[n]$, where $\text{sd}_{(i)}[n]$ is the complex with subchains of ← ($i \neq 0$!)

$$\{0\} < \{0,1\} < \dots < \{0,1,\dots,i-1\} < \{0,1,\dots,i-1,i+1\} < \dots$$

$$\dots < \{0,1,\dots,i-1,i+1,\dots,k\} < \dots < \{0,\dots,n\}$$

as simplices.

Thus $\text{sd}_{(i)}[n]$ is the of proper chains of subsets of $\{S \mid S \in \partial_i[n] \text{ or } S = [n]\}$

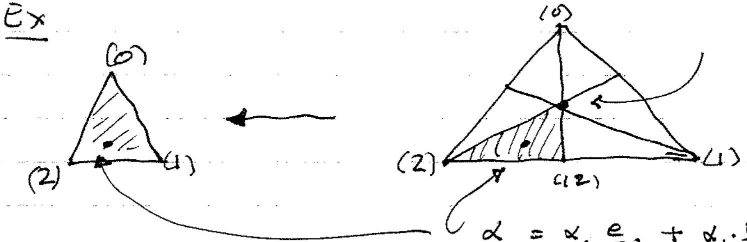
There are no very natural maps of simplicial complexes between K and its barycentric subdivision, but

Proposition There is a natural (convexity-preserving) homeomorphism

$$|\text{sd}\,K| \to |K|$$

of geometric realizations:

Ex



$$\alpha = \alpha_0\, \underline{\underline{e}}_2 + \alpha_1 \cdot \tfrac{1}{2}(\underline{\underline{e}}_1 + \underline{\underline{e}}_2) + \alpha_2 \cdot \tfrac{1}{3}(\underline{\underline{e}}_0 + \underline{\underline{e}}_1 + \underline{\underline{e}}_2)$$

$$= \tfrac{1}{3}\alpha_2 \cdot \underline{\underline{e}}_0 + (\tfrac{1}{2}\alpha_1 + \tfrac{1}{3}\alpha_2) \cdot \underline{\underline{e}}_1 + (\alpha_0 + \tfrac{1}{2}\alpha_1 + \tfrac{1}{3}\alpha_2) \cdot \underline{\underline{e}}_2$$

$\alpha_0 + \alpha_1 + \alpha_2 = 1 \Rightarrow$
$\tfrac{1}{3}\alpha_2 + (\tfrac{1}{2}\alpha_1 + \tfrac{1}{3}\alpha_2) + (\alpha_0 + \tfrac{1}{2}\alpha_1 + \tfrac{1}{3}\alpha_2) = 1$

(set of proper subsets of [n])

More generally, if S is a simplex of $\mathrm{sd}_i [n]$, let $\underline{\underline{e}}_S \in |\mathrm{sd}_i\, \Delta^n| \subset \mathrm{Maps}(2_*^{[n]}, I)$ be the function $\underline{\underline{e}}_S(T) = \delta_{S,T}$; such elements form a basis for $\mathrm{Maps}(2_0^{[n]}, \mathbb{R})$. [The standard basis elements $\underline{e}_i \in \mathrm{Maps}([n], I) = |\Delta^n|$ can be defined similarly, ie $\underline{e}_i(k) = \delta_{i,k}$.]

With this notation,

$$\underline{\underline{e}}_S \mapsto \frac{1}{\# S} \sum_{i \in S} \underline{e}_i = \text{average of } \{\underline{e}_i \mid i \in S\}$$

defines a homeomorphism

$$|\mathrm{sd}\, \Delta^n| \xrightarrow{\cong} |\Delta^n|,$$

which extends to a homeomorphism, simplex by simplex, of $|\mathrm{sd} K|$ with $|K|$.

By [Rotman 2.9 p37], the diameter of a simplex in the barycentric subdivision of $|\Delta^n|$ is

less than or equal to

$$\frac{n}{n+1} \cdot (\text{diameter } \Delta^n).$$

More generally [Rotman 6.15 p116], the mesh of a (geometric) simplicial complex is the supremum of the diameters of its simplices. It follows that

$$\boxed{\text{mesh}\, |\mathrm{sd} K| \leq \frac{n}{n+1}\, \text{mesh}\, |K|}$$

for any n-dimensional simplicial complex

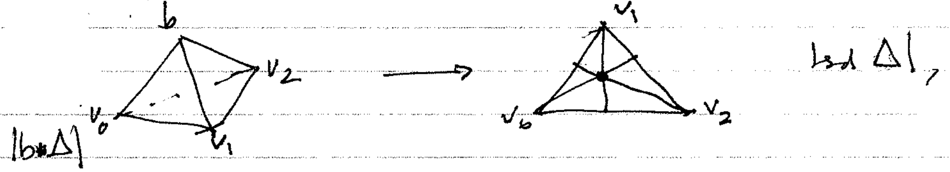
[check: $\frac{x}{1+x}$ is increasing for $x > 0$, by calculus.]

and hence that any geometric simplicial complex is homeomorphic to one, all of whose simplices have diameter $\leq$ any given $\epsilon > 0$.

(Proof, $\left(\frac{n}{n+1}\right)^k \to 0$ as $k \to \infty$!)

Remark A geometric simplex $|\Delta|$ is the convex hull of its set V of vertices, so adjoining a new vertex b, linearly independent of the elements of V, defines a new geometric simplex $|b*\Delta|$ (of one dimension higher) as its convex hull.

Projecting b to the barycenter $\{v_i \mid v_i \in V\} = \frac{1}{n}\sum v_i$ squashes $|b*\Delta|$ onto $|sd\,\Delta|$ in an intuitively satisfying way!



cf [Rotman Ch 4 p73] (which allows us to think of sd Δ as an (n+1)-dim'l cx if that would be convenient.)

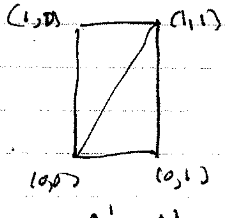
§ Products ④.4

The product $K\times L$ of simplicial complexes K, L has the product $V(K)\times V(L)$ of their

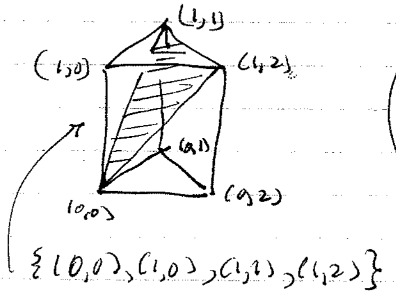
vertex sets, with the dictionary order, as its set of vertices. A simplex of $K\times L$ is a collection $\{(x_0,y_0),(x_1,y_1),\ldots,(x_n,y_n)\}$ of ordered pairs of vertices, such that

$\{x_0\le x_1\le \ \le x_n\}$ and $\{y_0\le\ldots\le y_n\}$ are simplices of K and L, respectively. Note that in this definition, x's and y's can be repeated:

Example The prism $\Delta^1\times\Delta^n$ $(\Delta^1=[0,1])$



$\Delta^1\times\Delta^1$



can be written in the form $[v_0,\ldots,v_{i-1},w_{i-1},\ldots,w_k]$

(k dim'l) has simplices of the form

$\{(0,0),\ldots,(0,i-1);(1,i-1),\ldots,(1,k-1)\}$, $0\le i<k\le n+1$

If we write $[v_0,\ldots,v_k]$ for the simplices of $0\times\Delta^n$ and $[w_0,\ldots,w_k]$ for those of $1\times\Delta^n$ then the simplices of $\Delta^1\times\Delta^n$

§ General simplicial objects

Abstract simplicial complexes have many uses, eg in combinatorics, but for many purposes they are more conveniently regarded as special cases of the more general class of simplicial objects. This involves a reinterpretation of the face and degeneracy operators (p v-vi above), in terms of the simplicial category Δ:

Definition, The objects of Δ are the finite ordered sets $[n] = \{0, .., n\}$, with nondecreasing (monotone) ~~functions~~ $f : [m] \to [n]$, $(i < j \Rightarrow f(i) \leq f(j))$ as maps.

Example A simplicial complex K defines a contravariant functor

$$\Delta \ni [n] \mapsto \mathrm{Maps}_{(\mathrm{Cxes})}(\Delta^n, K) := K[n] \in (\mathrm{Sets}),$$

often written

$$K \mapsto K[\cdot] : (\mathrm{Simp.\ Cxes}) \longrightarrow \Delta^{op}(\mathrm{Sets}).$$

The category $\Delta^{op}(\mathrm{Sets})$ (of simplicial sets) is thus the category of contravariant functors from finite ordered sets to sets; similarly, the category $\Delta^{op}(\mathrm{Ab})$ of simplicial abelian groups is the category of (contravariant) functors (eg $A[\cdot]$) from Δ to ($\mathbb{Z}$-modules).

The functor $S \mapsto \mathbb{Z}[S]$ from sets to (free) abelian groups, for example, thus defines a functor

$$\Delta^{op}(\mathrm{Sets}) \longrightarrow \Delta^{op}(\mathrm{Ab}).$$

—//—

Geometric realization $K \mapsto |K|$ is a functor from simplicial complexes to topological spaces.

so the composition

$X[n] : [n] \mapsto |\Delta^n| \mapsto \mathrm{Maps}(|\Delta^n|, X) \in (\mathrm{Sets})$

defines a functor

$$X[\bullet] : (\mathrm{Spaces}) \longrightarrow \Delta^{op}(\mathrm{Sets})$$

which associates to a space X, a purely combinatorial object, sometimes called the set of <u>singular simplices</u>.

<u>Proposition</u> This functor respects products:

$$(X \times Y)[\bullet] = X[\bullet] \times Y[\bullet] ,$$

(where the product on the right is the natural construction for categories of functors to categories (like (Sets)) with products). That is,

$$\begin{array}{c} \mathrm{Maps}(|\Delta^n|, X \times Y) \cong \mathrm{Maps}(|\Delta^n|, X) \times \mathrm{Maps}(|\Delta^n|, Y) \\ \shortparallel \\ (X \times Y)[n] \cong X[n] \times Y[n] . \end{array}$$

<u>Remark</u> There is a product-preserving geometric realization functor

$$X[\bullet] \longmapsto |X| := \left(\coprod_{n \geq 0} |\Delta^n| \times X[n] \right) / (\text{identifications})$$

constructed by gluing geometric simplices together using the face and degeneracy operators on $X[\bullet]$. (Modulo some fine print about the compactly generated topology for infinite complexes) this construction satisfies

$$|X[\bullet] \times Y[\bullet]| \cong |X[\bullet]| \times |Y[\bullet]| .$$

<u>Note</u> that if the simplicial set $X[\bullet]$ is constructed as above from a topological space X (so $X[n] = \mathrm{Maps}(|\Delta^n|, X)$) then there is an evaluation map ;

$$\begin{tikzcd} {|X[\bullet]|} \arrow[rrd, dashed] & = & \left(\coprod_{n \geq 0} |\Delta^n| \times \mathrm{Maps}(|\Delta^n|, X) \right] / (\text{relations}) \arrow[d] \\ & & X \end{tikzcd}$$

which (in reasonable cases) is a homotopy equivalence, as we shall eventually see.

Ex If $\mathcal{C}$ is a (small) category, let $\mathcal{C}[0]$ denote its set of objects, and $\mathcal{C}[1]$ its set of maps; then there are

$$s, t \text{ (= source, target) } : \mathcal{C}[1] \rightrightarrows \mathcal{C}[0],$$

ie $s(x \xrightarrow{f} y) = x$, $t(x \xrightarrow{f} y) = y$, as well as an identity map $x \mapsto (x \xrightarrow{1_x} x) : \mathcal{C}[0] \longrightarrow \mathcal{C}[$

More generally, if $\mathcal{C}[n]$ denote the set of composable maps

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} x_n$$

in $\mathcal{C}$, then composing two consecutive maps in such a string defines maps $\mathcal{C}[n] \longrightarrow \mathcal{C}[n-1]$, while inserting identity maps $\mathcal{C}[n] \longrightarrow \mathcal{C}[n+1]$ define maps which can be easily seen to satisfy the degeneracy and face map conditions.

(of categories to the category $\Delta^{op}(\mathrm{Sets})$)

A (small) category thus defines a simplicial set $\mathcal{C}[\circ]$, characterized by the property that

$$\begin{tikzcd} & \mathcal{C}[2] \arrow[dl] \arrow[dr] & \\ \mathcal{C}[1] \arrow[dr, "s"'] & & \mathcal{C}[1] \arrow[dl, "t"] \\ & \mathcal{C}[0] & \end{tikzcd} \qquad \mathcal{C}[2] = \mathcal{C}[1] \times_{\mathcal{C}[0]} \mathcal{C}[1]$$

is a fiber product (and, more generally, $\mathcal{C}[n] = \mathcal{C}[1] \times_{\mathcal{C}[0]} \cdots \times_{\mathcal{C}[0]} \mathcal{C}[1]$ is an n-fold iterated fiber product).

Ex A group G can be regarded as a category with one object $* = G[0]$, and $G[1] = G$ as its set of (endo)morphisms. The classifying space of the resulting simplicial set is called the classifying space

$$BG = |G[\bullet]|$$

of the group. Ex's $B\mathbb{Z} = \mathbb{T}$

$B\mathbb{Z}_2 = \mathbb{RP}^\infty$

note, this is a genuinely ∞-dim'l object, not a finite cx.

More generally, a group action $G \times X \to X$ defines a category $[X/G]$ with X as its objects, and $\mathrm{Maps}(x,y) = \{ g \in G \mid gx = y \}$. For example, the action of G on itself by multiplication defines a ~~category~~ $[G/G]$ and a functor

$$[G/G] \to [*/G]$$

Taking classifying spaces defines a space

$$|[G/G]| := EG$$

with a free G-action, such that the quotient is $|[*/G]| = BG$. It follows from covering space theory that

$$\pi_i BG = 0 \qquad i \neq 1$$
$$= G \qquad i = 1 .$$

A similar construction associates to a <u>topological</u> group G, a <u>topological</u> category $[G/G]$ whose

in a natural sense

geometric realization EG is still) contractible; but the classifying space $BG = \|[\ /G]\|$ is <u>not</u> necessarily concentrated in one degree.

If X is a space with G action, the quotient map

$$\begin{tikzcd}
(X \times EG)/G \arrow[r] \arrow[d, equal] & X \times_G \mathrm{pt} \arrow[d, equal] \\
X/\!/G \arrow[r] & X/G
\end{tikzcd}$$

defines a kind of homotopy-theoretic resolution of the geometric quotient.

i)

Part V | Basics of Homological Algebra

§ Chain Complexes (6.1)

If R is a (commutative, unital) ring, eg $\mathbb{Z}$ or a field, the category $(R\text{-Mod})_*$ of graded R-modules (eg $(R\text{-Mod})_* = (Ab_*)$ = the category of graded abelian groups if $R = \mathbb{Z}$) was defined in §3.2 p xi) – xiii) above.

Chain complexes (over R) are graded R-modules with extra structure, ie graded objects C_* together with degree decreasing homomorphisms

$$\partial^C_i : C_i \to C_{i-1} \quad \in \operatorname{Hom}_R(C_*, C_{*-1})$$

such that the compositions $\partial^C_i \circ \partial^C_{i+1} = 0$

$$\begin{tikzcd} C_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[rr, bend right, "0"'] & C_i \arrow[r, "\partial^C_i"] & C_{i-1} \end{tikzcd}.$$

of consecutive "differentials" are zero.

Ex. An exact sequence

$$\to E_{i+1} \xrightarrow{\alpha_{i+1}} E_i \xrightarrow{\alpha_i} E_{i-1} \to \cdots$$

ii).

is a chain complex, because

$$\ker \alpha_i = \text{image}\ \alpha_{i+1} \quad :$$

thus if $v \in E_{i+1}$, then $\alpha_{i+1}(v) \in \text{image}\ \alpha_{i+1} = \ker \alpha_i$,

so $\alpha_i(\alpha_{i+1}(v)) = (\alpha_i \circ \alpha_{i+1})(v) = 0$.

However, being a chain complex is less restrictive than being an exact sequence: the chain complex condition $\partial_i \circ \partial_{i+1} = 0$ implies only that $\text{image}\ \partial_{i+1} \subset \ker \partial_i$, not necessarily that they are equal. Consequently the homology groups

$$H_i(C) := \ker \partial_i / \text{image}\ \partial_{i+1} \in R\text{-Mo}$$

measure how far a chain complex is, from being exact.

Ex The chain complex

$$\begin{tikzcd} \underset{C_3}{0} \arrow[r] & \underset{C_2}{\mathbb{Z}} \arrow[r, "n"] & \underset{C_1}{\mathbb{Z}} \arrow[r, "\partial_1 = 0"] & \underset{C_0}{0} \end{tikzcd}$$

ie $\partial_2(k) = nk$ is multiplication by $n \in \mathbb{N}$

iii)

has homology groups

$$H_1 = (\mathbb{Z} = \ker \partial_1)/(n\mathbb{Z} = \text{image}\,\partial_2) = \mathbb{Z}_n$$

$$H_* = 0 \text{ otherwise}$$

<u>Definition</u> A homomorphism $(C_*, \partial^C_*) \xrightarrow{\phi_*} (D_*, \partial^D_*)$ of <u>chain complexes</u> is a homomorphism $\phi_* : C_* \to D_*$ of graded R-modules with the additional property that the chain

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+1} \arrow[r, "\partial^C_{i+1}"] \arrow[d, "\phi_{i+1}"] & C_i \arrow[r, "\partial^C_i"] \arrow[d, "\phi_i"] & C_{i-1} \arrow[r] \arrow[d, "\phi_{i-1}"] & \cdots \\
\cdots \arrow[r] & D_{i+1} \arrow[r, "\partial^D_{i+1}"] & D_i \arrow[r, "\partial^D_i"] & D_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

of diagrams <u>all</u> <u>commute</u>, ie

$$✩ \quad \forall i \quad \phi_{i-1}\partial^C_i = \partial^D_i \phi_i \,.$$

$\boxed{\text{Exercise}}$ A chain homomorphism ϕ_*, as above, induces a homomorphism

$$H(\phi)_* : H_*(C) \longrightarrow H_*(D)$$

iv)

of homology groups, such that

$$(C_*, \partial_*) \longmapsto H_*(C)$$

is a <u>functor</u>

$$(\underline{\text{Cxes}} \text{ of R-modules}) \to (\text{graded R-Modules}).$$

<u>Remark</u> As in § 3.2 p xi), the set of homomorphisms of <u>chain</u> <u>complexes</u> from (C_*, ∂^C_*) to (D_*, ∂^D_*) is an R-module.

[of the <u>shift</u> or <u>suspension</u> functor on the category of chain cxes]

<u>Definition</u> If (C_*, ∂_*) is a chain complex,

$$S^i(C_*, \partial^C_*) = (C_{*-i}, \partial^C_{*-i})$$

(obtained by shifting the grading) is again a chain complex. There is a graded module

$$\text{Hom}_k((C_*, \partial^C_*), (D_*, \partial^D_*))$$

$$:= \text{Hom}_{\text{Cxes}}(S^k(C_*, \partial^C_*), (D_*, \partial^D_*))$$

(the superscript here indicates a <u>negative</u> shift of subscript)

of chain homomorphisms of degree k. [We will see later that this graded group has a natural

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Ex A homomorphism $\alpha : A \to B$ of R modules defines a chain cx $0 \to A \xrightarrow{\alpha} B \to 0$ with $H_0 = \ker \alpha$, $H_1 = B/\text{image}\,\alpha := \text{coker}\,\alpha$. If R is a <u>field</u>, and the H_i are finite-dimensional, $\chi(\alpha) = \dim \ker \alpha - \dim \text{coker}\,\alpha$ is called (particularly in analysis) the <u>index</u> of α. It characterizes <u>Fredholm</u> operators in analysis, and is particularly important in the theory of elliptic PDO's.

differential ∂^D_{C*} of its own, making A into a chain complex, but we won't need that for a while).

<u>Lemma</u> Suppose (C_*, ∂^C_*) and (D_*, ∂^D_*) are chain complexes, and that $q_* : C_* \to D_{*+1}$ is a homomorphism of graded groups. Then

$$h_i := \partial^D_{i+1} \circ q_i + q_{i-1} \circ \partial^C_i$$

defines a homomorphism $h_* : (C_*, \partial^C_*) \to (D_*, \partial^D_*)$ of chain complexes such that

$$H(h)_* = 0 : H_*(C) \to H_*(D)$$

is trivial on homology.

<u>Proof:</u> To see that h_* is a homomorphism of chain complexes we need to check that

$$h_{i-1} \circ \partial^C_i = (\partial^D_i \circ q_{i-1} + q_{i-2} \partial^C_{i-1}) \circ \partial^C_i$$

$$= \partial^D_i \circ q_{i-1} \circ \partial^C_i \quad \text{equals}$$

$$\partial^D_i \circ h_i = \partial^D_i \circ (\partial^D_{i+1} \circ q_i + q_i \circ \partial^C_i)$$

$$= \partial^D_i \circ q_i \circ \partial^C_i .$$

To see that $H(h)_* = 0$, suppose that $z \in \ker \partial^C_i$. Then

$$h_i(z) = (\partial^D_{i+1} \circ q_i + q_{i-1} \partial^C_i) z$$

$$= \partial^D_{i+1}(q_i(z)) \in \text{image}\, \partial^D_{i+1},$$

i.e. $h_i(z) = 0$ in $H_i(D)$. ▨

Convention At this point I'll stop including the boundary operator in the notation, and simply write

$$\mathrm{Hom}_{cxs}(C_*, D_*)$$

for the R-module of chain homomorphisms from (C_*, ∂^C_*) to (D, ∂^D_*), etc. I'll also write

$$\mathrm{Hom}^0_{cxs}(C_*, D_*) =$$

$$\{h_* \in \mathrm{Hom}_{cxs}(C_*, D_*) \mid h_* = \partial^D_{*+1} \circ q_* + q_{*-1} \circ \partial^C_* \text{ for some } h_* : C_* \to D_{*+1}$$

vii)

Proposition ∃ factorization

$$\begin{tikzcd}
\mathrm{Hom}^0_{Cx}(C_*, D_*) \times \mathrm{Hom}_{Cx}(D_*, E_*) \arrow[r, "{h,\psi \mapsto \psi \circ h}"] \arrow[d, dotted] & \mathrm{Hom}_{Cx}(C_*, D_*) \times \mathrm{Hom}_{Cx}(D_*, E_*) \arrow[d] \\
\mathrm{Hom}^0_{Cx}(C_*, E_*) \arrow[r] & \mathrm{Hom}_{Cx}(C_*, E_*)
\end{tikzcd}$$

More precisely if $\psi : D_* \to E_*$ is a homomorphism of chain complexes, and $h_* = \partial^D_{*+1} \circ q_* + q_{*-1} \circ \partial^C_*$, then

$$\psi_* (\partial^D_{*+1} \circ q_* + q_{*-1} \circ \partial^C_*)$$

$$= \partial^E_* \circ (\psi_{*+1} \circ q_*) + (\psi_* \circ q_{*-1}) \circ \partial^C_* .$$

[Similarly for Hom × Hom^0 . . .] (HCx)

More precisely the category of chain cxes & homotopy classes of chain homomorphisms

Definition The homotopy category of chain complexes (of R-modules) has chain complexes as objects, with the R-module

$$\mathrm{Hom}_{Cx}(C_*, D_*) / \mathrm{Hom}^0(C_*, D_*)$$

$$:= \mathrm{Hom}_{HCx}(C_*, D_*)$$

as maps.

viii)

Note that by the preceding proposition, this is a category: that is, we have associative composition homomorphisms

$$\mathrm{Hom}_{HCx}(C_*, D_*) \times \mathrm{Hom}_{HCx}(D_*, E_*) \to \mathrm{Hom}_{HCx}(C_*, E$$

Note also that two chain homomorphisms

$$f_*, g_* : C_* \longrightarrow D_*$$

are equivalent in the homotopy category (or: are chain homotopy equivalent) ⟺ ∃ a chain homotopy

$$q : C_* \to D_{*+1}$$

such that $f_* - g_* = \partial^D_{*+1} \circ q_* + q_{*-1} \circ \partial^C_*$.

Proposition ∃ factorization

$$\begin{tikzcd}
(\text{Chain Cxes}) \arrow[r, "H_*"] \arrow[dr] & (\text{graded modules}) \\
& \mathrm{Ho}(\text{Chain Cxes}) \arrow[u, dotted]
\end{tikzcd}$$

of the homology functor through the homotopy category.

ix)

[For homology $f \mapsto H(f)$.

$$\mathrm{Hom}_{cx}(C_*, D_*) \longrightarrow \mathrm{Hom}_{R\text{-Mod}}(H_*(C), H_*(D))$$

kills the submodule $\mathrm{Hom}^0_{cx}(C_*, D_*)$.]

Example Recall [§4.1 p v-vii] the simplicial chain functor

$$\begin{aligned}(\text{Simplicial cxs}) &\longrightarrow (R\text{-Mod})_* \\ K &\longmapsto C_*(K,R)\end{aligned}$$

where for ex $C_k(K,\mathbb{Z})$ is the free abelian group generated by the k-simplices $[v_0, .., v_k]$ of K.

(with the convention that repeats in the v_i are allowed; they just imply that $[v_0, \ldots, v_j, \ldots, v_j, \ldots, v_k] = 0$.)

The face maps $\partial_i [v_0, .., v_k] \mapsto [v_0, \ldots, \hat{v}_i, \ldots, v_k]$ define differentials

$$\partial_k^{C(K)} : C_k(K) \longrightarrow C_{k-1}(K)$$

x)

by $$\partial_k^{C(K)} [v_0, \ldots, v_k] = \sum (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_k]$$

making $(C_*(K), \partial_*^{C(K)})$ into a chain cx:

$$\partial_{k-1}^{C(K)} \circ \partial_k^{C(K)} = 0$$

Proof See [Rotman Thm 4.6 p65]. If $f : K \to K'$ is a map of simplicial complexes, then

$$\begin{aligned}\partial_i f [v_0, \ldots, v_k] &= \partial_i [f(v_0), \ldots, f(v_k)] \\ &= [f(v_0), \ldots, \widehat{f(v_i)}, \ldots, f(v_k)] \\ &= f [v_0, \ldots, \hat{v}_i, \ldots, v_k] = \partial_i f,\end{aligned}$$

so the simplicial chain group functor lifts to a functor from simplicial complexes to chain complexes (of R-modules ...). Composing with H_* defines a functor from simplicial complexes to $(R\text{-Mod})_*$, called the (simplicial) homology $H_*(K,R) := H_*(C(K), \partial)$ of K.

Proposition The (simplicial) homology groups of a simplex Δ^n are *zero* in positive dimensions:

$$H_*(\Delta^n, \mathbb{Z}) = 0, \quad * > 0.$$

Proof, Following [Rotman Th 4.19 p 72], using the identification of Δ^{n+1} as $b * \Delta^n$ as in (§4.3 p xvii):

I claim that the homomorphisms

$$c_k : C_k(b * \Delta^n) \longrightarrow C_{k+1}(b * \Delta^n)$$

defined on generators by

$$c_k[v_0, \dots, v_k] = [b, v_0, \dots, v_k]$$

satisfy

$$\boxed{\partial^{b\Delta}_{k+1} c_k + c_{k-1}\partial^{b\Delta}_k = \text{identity}_{b\Delta} \text{ if } k > 0 :}$$

for $\partial^{b\Delta}_{k+1} c_k[v_0, \dots, v_k] + c_{k-1}\partial^{b\Delta}_k[v_0, \dots, v_k] =$

$$= \partial^{b\Delta}_{k+1}[b, v_0, \dots, v_k] \qquad \Big(= \sum_{0 \le i \le k} (-1)^i [b, v_0, .., \hat{v}_i, v_k]$$

$$= [v_0, \dots, v_k] - \sum_{0 \le i \le k} [b, v_0, \dots, \hat{v}_i, \dots, v_k]$$ cancels

$$= [v_0, \dots, v_k] \;;$$

at least, if none of the vertices of $[v_0, \dots, v_k]$ is b!

But if, for example v_i, equals b, then

$$c_k[v_0, \dots, v_k] = 0,$$

while

$$c_{k-1}\partial_k[v_0, \dots, v_k] = (-1)^i c_{k-1}[v_0, \dots, \hat{v}_i, \dots, v_k]$$

$$= (-1)^i [b, v_0, \dots, \hat{v}_i, \dots, v_k]$$

$\longleftrightarrow$ shift i places

$$= [v_0, \dots, v_k].$$

Consequently, the identity map of

$$(C_*(\Delta^{n+1}), \partial^{\Delta^{n+1}}_*) \cong (C_*(b * \Delta^n), \partial^{b\Delta^n}_*)$$

lies in $\mathrm{Hom}^0_{\mathrm{Cx}_*}(C_*(\Delta^{n+1}), C_*(\Delta^{n+1}))$ — or, in other words, it is chain homotopic to the zero map.

Thus $H_*(\mathrm{id}) = 0 : H_*(\Delta^{n+1}) \longrightarrow H_*(\Delta^{n+1})$,

so $H_*(\Delta^{n+1}) = 0$ if $* > 0$.

[$(\partial_{k+1}c_k + c_{k-1}\partial_k)[v_i] = [b] - [v_i]$ on $C_0(b * \Delta^n)$,
so if $* = 0$ this argument fails!]

Ex (Persistent homology)

If $i^r_s : X_r \to X_s$, $(r, s \in \mathbb{N})$ is a family of maps such that $i^s_s = \mathrm{id}_{X_s}$, $i^s_t \circ i^r_s = i^r_t$ if $s \le r \le t$,

then

$$H_*(\underline{X}, R) = \bigoplus_{r \geq 0} H_*(X_r, R) \in (R[T]\text{-Mod})_*,$$

where T acts on $H_*(X_r, R)$ as $(i^r_{r+1})_*$.

Ex. If $\epsilon > 0$ and $X_r := X(r\epsilon)$ then the Rips complexes define such a family.

Note also the useful fact that (when $R = \mathbb{F}$ is a field) the polynomial ring $\mathbb{F}[T]$ is a principal ideal domain, and hence that its modules are easily classified.

Exercise Show that $(C_*, \partial^C_*) \otimes (D_*, \partial^D_*)$

$:= (C_* \otimes D_*, \partial^{C\otimes D}_*)$ is a chain complex; where

$$\partial^{C\otimes D}_* = \partial^C_* \otimes 1 + (-1)^? 1 \otimes \partial^D_* ;$$

and, similarly, that $\mathrm{Hom}_{cx}(C_*, D_*)$ is a chain complex, with a similar differential.

(cf §6.3 p xviii)

x(U) ⑤.②

§ Singular homology

Definition The singular chain complex

$$S_n(X, \mathbb{Z}) := \mathbb{Z}[X[n]]$$

of a topological space X is constructed from the simplicial abelian group generated by X[·]. (cf §4 p $\overset{\text{vii}-\text{viii}}{\text{xix}-\text{xx}}$) by defining differentials

$$\partial_n^X = \sum_{0 \le i \le n} (-1)^i \partial_i$$

ix just as on p ix) above. [Note that this construction associates a chain complex to an arbitrary simplicial abelian group!]

$\boxed{\text{The singular homology } H_*(X, \mathbb{Z}) \text{ of } X \text{ is the homology of } (S_*(X), \partial_*^X)}$

more generally: $S_n(X, M)$

Elements of $S_n(X, \mathbb{Z})$ can thus be written as finite sums $\sum a_i \sigma_i$ with coefficients $a_i \in \mathbb{Z}$ (more generally, $a_i \in$ some R-module M), where $\overset{\text{the generator}}{\sigma_i} : |\Delta|^n \to X$ is a 'singular simplex'

xvj]

i.e. a continuous map

☆ A (convenient, standard) notational convention often denotes such a generator by something like

$$\sigma := \sigma \circ [v_0, \ldots, v_n] : |\Delta|^n \to X,$$

which allows us to write

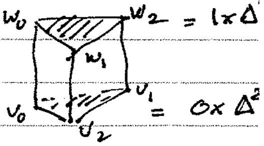
$$\partial_i \sigma = \sigma \circ [v_0, \ldots, \hat{v}_i, \ldots, v_n] : |\Delta^{n-1}| \to X$$

Example (following §4.4 p xviii)

The prism operator

$$P_* : S_*(X) \to S_{*+1}(I \times X)$$

is defined by

$I \times \Delta^2 =$ 

$$P_k(\sigma \circ [v_0, \ldots, v_k) :=$$

$$\sum_{0 \le i \le k} (-1)^i (1_I \times \sigma) \circ [v_0, .., v_i, w_i, \ldots, w_k]$$

interpreted as a singular simplex $|\Delta^{n+1}| \to I \times |\Delta^n| \xrightarrow{1_I \times \sigma} I \times$

Note that, unlike the simplicial chains defined in §4.1, we do not impose relations coming from permutations of the vertex labels. Simplicial chains are defined by their geometric images, whereas singular chains are defined by maps.

<u>Proposition</u> $\left(\partial^{I\times X}_{k+1}\circ P_k + P_{k-1}\circ\partial^X_k\right)(\sigma\circ[v_0,\dots,v_k])$

$= (1_I\times\sigma)[w_0,\dots,w_k] - (1_I\times\sigma)[v_0,\dots,v_k]$

$\in S_{k+1}(I\times X)$.

<u>Proof</u>; By the definition above we have

$$\left(\partial^{I\times X}_{k+1}\circ P_k\right)(\sigma\circ[v_0,\dots,v_k]) =$$

$$\sum_{0\le i\le k}(-1)^i(1_I\times\sigma)\partial_{k+1}[v_0,\dots,v_i,w_i,\dots,w_k]$$

$$= \sum_{j\le i}(-1)^j(-1)^i[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_k]$$

$$+ \sum_{j\ge i}(-1)^{j+1}(-1)^i[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_k]$$

NB $(j+1)^{\text{th}}$ term

Here the $i=j$ terms appear with a + sign in the first sum, and with a − sign in the second, and hence <u>cancel</u>; <u>except</u> for the <u>first</u> term in the <u>first</u> sum, and the <u>last</u> term in the <u>second</u> sum, which add up to

$$(1_I\times\sigma)[w_0,\dots,w_k] - (1_I\times\sigma)[v_0,\dots,v_k].$$

On the other hand

$$P_{k-1}(\partial^X_k(\sigma\circ[v_0,\dots,v_k])) = \sum(-1)^j P_k(\sigma\circ[v_0,\dots,\hat{v}_j,\dots,v_k])$$

$$= \sum(-1)^j(-1)^i(1\times\sigma)\circ[v_0,\dots,v_i,w_i,\dots,\hat{w},\dots,w_k]$$

$$+$$

$$\sum(-1)^j(-1)^{i-1}(1\times\sigma)\circ[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_k]$$

which cancels the remaining terms. ∎

⊛ <u>Theorem</u> If $F : I\times X \to Y$ is a homotopy (from $f_0 = F|0\times X$ to $f_1 = F|1\times X$) then the composition

$$S(F)_*\circ P_* : S_*(X) \to S_{*+1}(Y)$$

is a chain homotopy between $S(f_0)_*$ and $S(f_1)_*$.

Consequently

$$H(f_0)_* = H(f_1)_* : H_*(X) \to H_*(Y)$$

define the <u>same</u> homomorphism of singular homology groups.

<u>Corollary</u> (to the proof): The singular chain complex <u>defines</u> a functor $X \mapsto (S_*(X), \partial^X_*)$ from the homotopy category of (pointed) spaces (§3.5 p xiii) to the homotopy category of chain complexes (§5.1 p

§5.3 Locality of the singular complex

The singular chain complex has important §4.3 p xvi) locality properties. Barycentric subdivision decomposes a geometric simplex nicely into a union of ~~simplices~~ of smaller diameter, and this section sketches a version of this construction in the context of chain complexes.

Suppose that $\{U_\alpha \subset X\}$ is a collection of subsets of X, such that the union

$$\bigcup U_\alpha^\circ = X$$

of their interiors covers X. If

$$\mathcal{U}[n] := \{\sigma \in X[n] \mid (\exists \alpha)\ \sigma(|\Delta^n|) \subset U_\alpha^\circ\}$$

is the set of singular simplices in X with image lying in the interior of some member

of the covering family, then $\mathcal{U}[\cdot]$ is a simplicial set: if $\sigma : |\Delta^n| \to U_\alpha^\circ$ and $\delta : |\Delta^m| \to |\Delta^n|$ is a composition of face and degeneracy operators, then $\sigma \circ \delta \in \mathcal{U}[m]$. Let $(S_*(\mathcal{U}), \partial_*^{\mathcal{U}})$ be the associated chain complex: since $\mathcal{U}[\cdot] \subset X[\cdot]$, there is an obvious induced map of chain complexes.

<u>Proposition</u> The inclusion

$$i_* : (S_*(\mathcal{U}), \partial_*^{\mathcal{U}}) \longrightarrow (S_*(X), \partial_*^X)$$

is a chain homotopy equivalence.

<u>Proof</u> (cf Hatcher, Prop 2.21 p 119): we construct a chain homotopy inverse

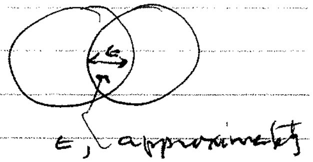
$$\rho_* : (S_*(X), \partial_*^X) \longrightarrow (S_*(\mathcal{U}), \partial_*^{\mathcal{U}}).$$

The argument uses Lebesgue's covering lemma from measure theory:

xx)

The supremum of such ϵ's is called the Lebesgue number of the cover

Proposition If $\{\mathcal{U}_\alpha\}$ is an open cover of a compact metric space X, then $\exists \epsilon > 0$ such that any subset $S \subset X$ of diameter $< \epsilon$ is contained in some member of the cover:



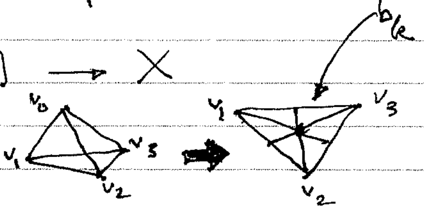
[The compact metric space in the argument below will be $|\Delta^n|$, not the arbitrary space X!]

Recall [§5.1 p xi] the construction

$$\beta_k(\sigma \circ [v_0, \ldots, v_k]) := \sigma \circ [b_k, v_0, \ldots, v_k]$$

which sends the singular simplex

$$\sigma : |\Delta^k| (= [v_0, \ldots, v_k]) \longrightarrow X$$



to the composition

$$|\Delta^{k+1}| \xrightarrow{b_k} |\Delta^k| \xrightarrow{\sigma} X$$

defined on vertices by $v_0 \mapsto \frac{1}{k}\sum_{1 \le i \le k} v_i$, $v_i \to v_{i-1}$ if $i > 0$.

xxii)

Beware Dangerous notational ambiguity! All these β are intended to use the same barycenter!

and which satisfies

$$(*) \quad \partial^X_{k+1}\beta_k + \beta_{k-1}\partial^X_k = 1_{S_k} : S_k(X) \to S_k(X)$$

(see note after proof)

$\boxed{\text{Rotman 6.12–6.14 p 114–116}}$

Proposition The inductive definitions

$$B_k = \beta_{k-1}(B_{k-1}\partial^X_k) : S_k(X) \to S_k(X)$$

$$T_k = \beta_k(1_{S_k} - B_k - T_{k-1}\partial^X_k) : S_k(X) \to S_{k+1}($$

(with B_0 = identity, $T_0 = 0$) yield a chain

$\boxed{\text{the barycentric subdivision homomorphism}}$

homomorphism B_*, chain homotopic

$$T_{k-1}\partial^X_k + \partial^X_{k+1}T_k = 1_{S_k} - B_k$$

to the identity map of $(S_*(X), \partial^X_*)$.

Note $1 - B_0 = 0$
$= T_{-1}\partial_0 + \partial_1 T_0$
$\phantom{= T_{-1}}\overset{\shortparallel}{0} \qquad \overset{\shortparallel}{0}$

Proof First of all, B_* is a chain map, by induction

$$\partial_k B_k = \partial_k \beta_{k-1} B_{k-1}\partial_k \qquad \text{(by definition)}$$

$$= (1_k - \beta_{k-1}\partial_k) B_{k-1}\partial_k \qquad \text{(by (*) above)}$$

$$= B_{k-1}\partial_k - \beta_{k-1}\partial_k B_{k-1}\partial_k$$

$$\hookrightarrow = \beta_{k-1} B_k \partial_{k-1} \circ \partial_k = 0 .$$

Similarly,

$$\partial_{k+1} T_k = \partial_{k+1} \beta_k (1_k - B_k - T_{k-1}\partial_k) \qquad \text{(by defn)}$$

$$= 1 - B_k - T_{k-1}\partial_k$$

$$- \beta_k \Big[\partial_k (1_k - B_k - T_{k-1}\partial_k) \qquad (\text{by } *)$$

$$\downarrow \text{ (by induction)}$$

$$-(1_{k-1} - B_{k-1} - T_{k-2}\underbrace{\partial_{k-1})\partial_k}_{\overset{\shortparallel}{0}}$$

(cancel) cancel

▨

Note In the definition $B_k(\sigma) = \beta_{k-1}(B_{k-1}\partial_k(\sigma))$, the operator β_{k-1} is meant to be understood as using (cf Hatcher, p121) the $\boxed{\text{barycenter of } \sigma}$ — not the barycenters of any of the simplices involved in $(B_{k-1}\partial_k)(\sigma)$. It is thus not quite distributive; otherwise we would have

$$B_k = \beta_{k-1}(B_{k-1}\partial_k) = \beta_{k-1}(\beta_{k-2}B_{k-2}\underbrace{\partial_{k-2})\partial_k}_{=0} \;!$$

Recall now [§4.3 pxvi] that the mesh of a geometric simplicial complex is the maximum diameter of one of

its simplices, and that barycentric subdivision reduces the mesh of an n-dimensional complex by at least a factor of $\frac{n}{n+1} < 1$. The mesh of a simplex bounds the mesh of its faces, so

Proposition [Rotman 6.15 p116]

$$\text{mesh}(B_n\sigma) \leq \frac{n}{n+1}\,\text{mesh}(\sigma).$$

Definition For any $\sigma \in X[n]$ there is a least integer $m(\sigma) \geq 0$ such that $B^{m(\sigma)} \in S_n(\mathcal{U})$: for $B^n(\sigma)$, $n \gg 0$, is a sum of simplices of diameter less than the Lebesgue number of the open cover $\{\bar{\sigma}^{-1}(U^0_\alpha)\}$ of the compact metric space $|\Delta^n|$.

We can now construct a chain homomorphism

$$\rho_* : S_*(X) \longrightarrow S_*(\mathcal{U})$$

which restricts to the identity on $S_*(\mathcal{U}) \subset S_*(X)$, together with a chain homotopy

$$D : i\circ\rho \simeq 1_{S_*(X)}$$

(where i is the inclusion, so $\rho\circ i = 1_{S_*(\mathcal{U})}$), as promised on p xix).

Definition Let T_* and B_* be as above. If $\sigma \in X[m]$ is a singular simplex of X, let

$$D(\sigma) = \sum_{0 \le j \le m(\sigma)-1} T B^j(\sigma)$$

(Note, if $\sigma \in \mathcal{U}[n]$ then $D(\sigma) = 0$!

[To simplify notation I'll stop subscripting operators with their dimensions.] The strategy of proof is to show that $\rho := 1 - (\partial D + D\partial)$ maps $S_*(X)$ to $S_*(\mathcal{U})$, and restricts to the

identity map on $S_*(\mathcal{U})$. It will thus automatically be a chain homomorphism, since

$$\partial\rho = \partial(1 - \partial D - D\partial) = \partial - \partial D\partial \quad \text{and}$$
$$\rho\partial = (1 - \partial D - D\partial)\partial = \partial - \partial D\partial .$$

Now

$$\partial D(\sigma) = \sum_{j \le m-1} \partial T B^j(\sigma)$$

which is a telescoping sum

$$= \sum (-T\partial B^j + B^{j+1} - B^j)\sigma$$

(since B^j is a chain map)

$$B^m\sigma - \sigma$$

$$-\sum (-1)^i T B^j(\partial_i\sigma)$$

while

$$D(\partial\sigma) = \sum (-1)^i \sum_{0 \le j \le m(\partial_i\sigma)-1} T B^j(\partial_i\sigma)$$

so

$$\sigma + \partial D\sigma + D\partial\sigma := \rho(\sigma) =$$

$$\sum_{m(\partial_i\sigma) \le j \le m(\sigma)-1} (-1)^i T B^j(\partial_i\sigma) + B^{m(\sigma)}\sigma \quad \in S_*(\mathcal{U})$$

(and $= 0$ if $\sigma \in \mathcal{U}[m]$). ▨

<u>Example</u> Suppose X is the union of two open sets U_0 and U_1, so there is a diagram

$$\begin{tikzcd}
 & U_0 \cap U_1 \arrow[dl, "i_0"'] \arrow[dr, "i_1"] & \\
U_0 \arrow[dr, "j_0"'] & & U_1 \arrow[dl, "j_1"] \\
 & U_0 \cup U_1 = X &
\end{tikzcd}$$

Then there is a short exact sequence

$$0 \to S_*(U_0 \cap U_1) \xrightarrow{S(i_0) \oplus S(i_1)} S_*(U_0) \oplus S_*(U_1) \xrightarrow{S(j_0) - S(j_1)} S_*(\mathcal{U}) \to 0$$

of chain complexes: for $S(i_0)$ and $S(i_1)$ are both obviously injective, and $u \oplus v \mapsto 0$ on the right iff u and v both come from $S_*(U_0 \cap U_1)$, where they are equal.

More generally, an exact sequence

$$0 \to A_* \to B_* \to C_* \to 0$$

of chain complexes is a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{i+1} \arrow[r, "\partial^A_{i+1}"] \arrow[d] & A_i \arrow[r, "\partial^A_i"] \arrow[d] & A_{i-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & B_{i+1} \arrow[r, "\partial^B_{i+1}"] \arrow[d] & B_i \arrow[r, "\partial^B_i"] \arrow[d] & B_{i-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{i+1} \arrow[r, "\partial^C_{i+1}"] \arrow[d] & C_i \arrow[r, "\partial^C_i"] \arrow[d] & C_{i-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which the columns are exact, and the rows are complexes — that is, the composition of consecutive homomorphisms is zero.

<u>Ex</u> 'A set inclusion $T \subset S$ induces a 1-to-1 homomorphism $\mathbb{Z}[T] \to \mathbb{Z}[S]$ of free abelian groups, with quotient $\mathbb{Z}[S]/\mathbb{Z}[T]$ isomorphic to the free abelian group on $S - T$; similarly, an inclusion $A \subset X$ of topological spaces defines an inclusion $A[\bullet] \subset X[\bullet]$ of simplicial sets, and an exact sequence

xxviii)

$$0 \to S_*(A) \to S_*(X) \to S_*(X)/S_*(A) := S_*(X,A) \to 0$$

of chain complexes of free abelian groups.

The homology groups of the complex $S_*(X,A)$ are the relative homology groups $H_*(X,A)$ of the pair; they are functorial under maps of pairs of spaces. The *excision axiom* asserts that if $Z \subset A$ is a subspace such that the closure $\bar{Z}$ of Z (in X) contained in the interior A° of A, then the map

$$(X-Z, A-Z) \to (X,A)$$

induces an isomorphism of (relative) homology groups.

If we write $B = X-Z$, then

$$(X-Z, A-Z) = (B, A \cap B)$$

xxix)

and the interior $B^\circ = X - \bar{Z}$ is the complement of the closure of Z (in X), so $\bar{Z} \subset A^\circ$ iff $X = A^\circ \cup B^\circ$: in other words, iff $\mathcal{U} = \{A, B\}$ is a cover of X as considered above.

Now $S_*(A \cap B)$ is a free sub-chain complex of $S_*(B)$, and the quotients

$$S_*(B)/S_*(A \cap B) \xrightarrow{\cong} S_*(\mathcal{U})/S_*(A)$$

are isomorphic as such, both being freely generated by singular simplices in B that do not lie in A. On the other hand the inclusion of $S_*(\mathcal{U})$ into $S_*(X)$ takes $S_*(A)$ to itself, so the chain homotopy equivalence constructed above induces a chain equivalence xxx)

$$S_*(\mathcal{U})/S_*(A) \xrightarrow{\cong} S_*(X)/S_*(A);$$

so under the hypotheses of the excision axiom,

$$H_*(X-Z, A-Z) = \cancel{H_*}(B, A\cap B) \xrightarrow{\cong} H_*(X,A)$$

is an isomorphism.

§5.4 The long exact sequences

The snake lemma [Rotman Ch 5 p93-96, lemma 5.5] asserts that if

$$0 \to A_* \xrightarrow{\alpha_*} B_* \xrightarrow{\beta_*} C_* \to 0$$

is an exact sequence of chain complexes (as in p xxviii above), then

$$\partial^{C,A}_n [c] := [\vec{\alpha}_n \partial^B_n \vec{\beta}_n (c)] : H_n(C) \to H_{n-1}(A)$$

is well-defined. Rotman gives a detailed proof, which I won't reproduce here. Moreover, his lemmas 5.6 - 5.7 show that this boundary homomorphism $\partial^{C,A}_n$ [unfortunately many things in algebraic topology are traditionally

xxxii)

denoted by some variant of ∂] fits in a long exact sequence

$$\cdots \to H_*(A) \xrightarrow{H(\alpha)_*} H_*(B) \xrightarrow{H(\beta)_*} H_*(C) \xrightarrow{\partial^{C,A}_*} H_{*-1}(A) \to \cdots$$

and that this construction is natural, in the sense that if

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r] \arrow[d, "\phi_*"] & B_* \arrow[r] \arrow[d, "\psi_*"] & C_* \arrow[r] \arrow[d, "\rho_*"] & 0 \\
0 \arrow[r] & A'_* \arrow[r] & B'_* \arrow[r] & C'_* \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of chain complexes and chain homomorphisms with exact rows, then the induced diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_*(A) \arrow[r] \arrow[d, "H(\phi)"] & H_*(B) \arrow[r] \arrow[d, "H(\psi)"] & H_*(C) \arrow[r, "\partial^{C,A}"] \arrow[d, "H(\rho)"] & H_{*-1}(A) \arrow[r] \arrow[d, "H(\phi)"] & \cdots \\
\cdots \arrow[r] & H_*(A') \arrow[r] & H_*(B') \arrow[r] & H_*(C') \arrow[r] & H_{*-1}(A') \arrow[r] & \cdots
\end{tikzcd}$$

is commutative. <u>IN THESE NOTES I WILL TAKE THESE FACTS AS GRANTED</u>, and focus on their (similarly for the FIVE LEMMA (R Th 5.10 iii, p98)) application to the short exact sequences

$(\cong S_*(X))$

(p xxvi): $0 \to S_*(\mathcal{U}_0 \cap \mathcal{U}_1) \to S_*(\mathcal{U}_0) \oplus S_*(\mathcal{U}_1) \to S_*(\mathcal{U}) \to 0$

xxxiii) and p xxviii) $0 \to S_*(A) \to S_*(X) \to S_*(X,A) \to 0$

constructed above: applying these lemmas from homological algebra, we have (natural) long exact

Mayer-Vietoris

$$\cdots \to H_*(U_0 \cap U_1) \to H_*(U_0) \oplus H_*(U_1) \to H_*(U_0 \cup U_1) \to H_{*-1}(U_0 \cap U_1) \to \cdots$$

and relative homology

$$\cdots \to H_*(A) \to H_*(X) \to H_*(X,A) \to H_{*-1}(A) \to$$

sequences.

The first of these is a categorification of Euler's

$$\chi(U_0 \cap U_1) + \chi(U_0 \cup U_1) = \chi(U_0) + \chi(U_1)$$

and the second is a categorification of

$$\chi(X) = \chi(A) + \chi(X/A)$$

(under suitable hypotheses on A).

xxxiv)

Ex long exact sequence of a triple (X, A, B),

eg use $(X/B)/(A/B) \simeq X/A$

§6 Reasonable categories of Spaces

6.1 Cellular chains

The construction of a functor satisfying the ES axioms involves many new and possibly unfamiliar ideas, so it is useful to look back and consolidate some of the material we have discussed up till now.

In §3.8 (p xxxii) a *cell complex structure* on a space X (≠∅!) was defined informally as *filtration*

$$X_0 \subset X_1 \subset \dots \subset X_k := sk_k X \subset \dots$$

of X by subspaces, constructed inductively

$$X_{n+1} = X_n \cup_{\amalg \alpha_i} \left(\amalg B^{n+1}_{(i)} \right) \quad , \; i \in I(n+1)$$

by attaching (n+1-dimensional closed Euclidean balls $B^{n+1}_{(i)}$, also called *cells*) along *attaching maps*

$$\alpha_i : S^n_{(i)} = \partial B^{n+1}_{(i)} \to X_n .$$

Since X_0 is zero-dimensional, it is natural to choose one of its components as a basepoint, and we shall do so, and regard cell complexes as base-*pointed* spaces.

Cell complexes are the first topic discussed in Hatcher [Ch 0 p1-20], with more detail in an appendix (p519-528) and they are the topic of Ch8 of Rotman; ~~that~~ what's said here is adequate for *finite* cell complexes, but complexes with infinitely many cells require more

* care. In these lectures I will operate in safe mode and restrict, by and large, to the homotopy category of *finite* pointed *cell complexes*.

* cf. local finiteness condition (3) in Rotman p198

In general, the *reasonable* category of topological spaces consists of *pointed* spaces *homotopy equivalent* to a cell (a.k.a CW) complex, cf *Hatcher A.11 p528*

iii)

Definition If $i \in I(n+1)$, $k \in I(n)$ index cells in X_n and X_{n+1} respectively, let

$$\mathbb{Z} \ni \text{degree of } \alpha_{i,k} : H_n(S^n_{(i)}) \to H_n(S^n_{(k)})$$

be the degree of the composition .

$$\begin{tikzcd}
\alpha_{i,k} : S^n_{(i)} \arrow[r, "\alpha_i"] & X_n \arrow[r] & X_n / X_{n-1} \arrow[d, equal] & \\
 & & \bigvee_{j \in I(n)} S^n_{(j)} \arrow[r, "\pi_k"] & S^n_{(k)}
\end{tikzcd}$$

(where α_i is an attaching map, and π_k collapses all but the k^{th} sphere in its domain. Let $C_k(X)$ be the free abelian group generated by the k-cells of X; then

$$[B^{n+1}_{(i)}] \mapsto \sum_{k \in I(n)} \deg(\alpha_{i,k}) \, [B^n_{(k)}] :$$

$$\partial^X_{n+1} : C_{n+1}(X) \to C_n(X)$$

defines the complex of cellular chains of X.
[Hatcher, §2.2 p140; Rotman Ch 8 p 213]

iv)

This requires us to verify that this *is*, in fact, a chain complex, ie that $\partial^X_n \partial^X_{n+1} = 0$; the point is that

$$C_n(X) = H_n(X_n, X_{n-1})$$

by excision, and that we can therefore use the existence of singular homology in the proof. The long exact sequences for the pairs (X_n, X_{n-1}) and (X_{n-1}, X_{n-2}) define the homomorphisms in the diagram

$$\begin{tikzcd}
{} \arrow[r] & H_n(X_n, X_{n-1}) \arrow[r, "\partial"] \arrow[drr, "d"'] & H_{n-1}(X_{n-1}) \arrow[r] \arrow[d, equal] & \cdots \\
 & \cdots \arrow[r] & H_{n-1}(X_{n-1}) \arrow[r, "i"'] & H_{n-1}(X_{n-1}, X_{n-2})
\end{tikzcd}$$

which expresses the chain boundary operator d_n as the composition $d = i \circ \partial$. But then we can write $d_n \circ d_{n+1}$ as the composition:

v)

$$\begin{tikzcd}
H_{n+1}(X_{n+1}, X_n) \arrow[r, "\partial"] & H_n(X_n) \arrow[r, "i"] & H_n(X_n, X_{n-1}) \arrow[dl, "\partial'"'] \\
& H_{n-1}(X_{n-1}) \arrow[r, "i'"] & H_{n-1}(X_{n-1}, X_{n-2})
\end{tikzcd},$$

which contains two consecutive maps

$$\rightarrow H_n(X_n) \xrightarrow{i} H_n(X_n, X_{n-1}) \xrightarrow{\partial'} H_{n-1}(X_{n-1})$$

in the long exact sequence for the pair (X_n, X_{n-1}), whose composition is therefore zero.

THEOREM The homology of the complex of cellular chains of X is isomorphic to the homology of X.

Proof, (following Rotman Ch 8, ~ p 213-215 (and before him, Dold); with some simplifications, in that I will assume that X is a finite cell cx!)

We need a preliminary

vi)

Lemma (R 8.35 p 213) For a finite cell cx $X = \{X_n\}$ as above, and integers p, q, n:

1) $q \geq n$ or $n > p \Rightarrow H_n(X_p, X_q) = 0$

2) $q \geq n \Rightarrow H_n(X, X_q) = 0$

3) $n > q \Rightarrow H_n(X, X_q) \cong H_n(X_{n+1}, X_q)$

Proof:

1) : by induction on $p - q \geq 0$: Trivial if $p - q = 0$.

If $p - q > 0$ consider exact sequence of the triple (X_p, X_{q+1}, X_q),

$$\cdots \rightarrow H_n(X_{q+1}, X_q) \rightarrow H_n(X_p, X_q) \rightarrow H_n(X_p, X_{q+1})$$

concentrated in degree $q+1$, hence $= 0$

zero by the inductive hypothesis: $p - q > p - (q+1)$

Exactness $\Rightarrow$ middle term is zero.

2) follows from 1), in the case the filtration is finite, because

$$H_n(X, X_q) = H_n(X_p, X_q) \text{ for some } p \gg 0.$$

3) Consider the exact sequence of the triple (X, X_{n+1}, X_q):

$$\begin{tikzcd}
\cdots H_{n+1}(X, X_{n+1}) \arrow[r] & H_n(X_{n+1}, X_q) \arrow[r] & H_n(X, X_q) \arrow[dl, bend left] & \boxed{\therefore \text{ iso}} \arrow[l] \\
{=0 \text{ by } (2)} & & H_n(X, X_{n+1}) \\
& & {=0 \text{ by } (2)}
\end{tikzcd}$$

<u>Proof</u> of the theorem (Rotman p 214):

Suppose $n-1 > k$, and consider the map of exact sequences induced by the map

$$(X_{n+1}, X_n, \emptyset) \xrightarrow{\lambda_{n+1}} (X_{n+1}, X_n, X_k)$$

of triples: we have maps

$$\begin{tikzcd}
\arrow[r] & H_{n+1}(X_{n+1}, X_n) \arrow[r, "\partial_{n+1}"] \arrow[d, "="] & H_n(X_n) \arrow[r] \arrow[d, "\lambda_n"] & \cdots \\
\arrow[r] & H_{n+1}(X_{n+1}, X_n) \arrow[r, "\partial'_{n+1}"] & H_n(X_n, X_k) \arrow[r] & \cdots ,
\end{tikzcd}$$

(ie $\partial'_{n+1} = \lambda_n \circ \partial_{n+1}$) as well as maps

$$\begin{tikzcd}
& H_n(X_n) \arrow[dl, "\lambda_n"'] \arrow[dr, "j_n"] & \\
H_n(X_n, X_k) \arrow[rr, "j_n"] & & H_n(X_n, X_{n-1})
\end{tikzcd}$$

which fit together to define

$$\begin{tikzcd}
H_{n+1}(X_{n+1}, X_n) \arrow[r, "\partial_n"] \arrow[d, "\partial'_{n+1}"] \arrow[dr, bend left=60, "d_{n+1}"] & H_n(X_n) \arrow[d, "j_n"] \arrow[dl, "\lambda_n"'] \\
H_n(X_n, X_k) \arrow[r, "j_n"'] & H_n(X_n, X_{n-1})
\end{tikzcd}$$

and similarly (from $\lambda_n : (X_n, X_{n-1}, \emptyset) \to (X_n, X_{n-1}, X$

$$\begin{tikzcd}
H_n(X_n, X_{n-1}) \arrow[r, "\partial'_{n-1}"] \arrow[dr, "d_n"'] & H_{n-1}(X_{n-1}, X_k) \arrow[d, "i_{n-1}"] \\
& H_{n-1}(X_{n-1}, X_{n-2})
\end{tikzcd}$$

which fit together in a diagram

$$\begin{tikzcd}
& & & 0 \arrow[d, equal] & \text{(part 1) of the lemma} \arrow[l] \\
& H_{n+1}(X_{n+1}, X_n) \arrow[d, "\partial'_{n+1}"] \arrow[dr, "d_{n+1}"] & & H_{n-1}(X_{n-2}, X_k \arrow[d] \\
0 \arrow[r] & H_n(X_n, X_k) \arrow[r, "j_n"] \arrow[d] & H_n(X_n, X_{n-1}) \arrow[r, "\partial'_n"] \arrow[dr, "d_n"'] & H_{n-1}(X_{n-1}, X_k) \arrow[d, "i_{n-1}"] \\
\text{(part 2) of the lemma} \arrow[u] & H_n(X_{n+1}, X_k) \arrow[d] & & H_{n-1}(X_{n-1}, X_{n-2} \\
& H_n(X_{n+1}, X_n) = 0 \ (\text{by part (1) of the lemma}) & &
\end{tikzcd}$$

with exact columns & central row.

ix)

Now by the exactness of the left column

$$\begin{aligned} H_n(X_{n+1}, X_k) &= H_n(X_n, X_k)/\text{image}\,\partial_{n+1} \\ \| \quad & \\ H_n(X, X_k) & \ \left(\text{by part 2 of the lemma}\right) \end{aligned}$$

but $\underline{j_n} : H_n(X_n, X_k) \to H_n(X_n, X_{n-1})$ is <u>injective</u> (exactness of the middle row), so

$$H_n(X, X_k) = \underset{\substack{\| \\ \ker \partial'_n}}{(\text{image } j_n)} \Big/ \overset{= \text{image } d_{n+1} \text{ by definition.}}{(\text{image } j_n \partial_{n+1})}$$

by exactness of middle row.

But i_{n-1} is 1-1 by exactness of the right column, so $\ker \partial'_n = \ker i_{n-1}\partial'_n \overset{\text{definition}}{=} \ker {}_n$, so

$$\begin{aligned} H_n(X, X_k) &= \ker d_n / \text{image } d_{n+1} \\ &= H_n(\text{Cx of cellular chains}) \end{aligned}$$

if $n-1 > k$, eg $k=0$!

$\boxed{6.2} \longrightarrow \boxed{6.2.1}$

<u>Remarks</u>. This often provides a very efficient way to calculate the homology of a space.

Note that simplicial chains are a special kind of cellular chain!

x)

For example, real projective space

$$\mathbb{RP}^n = \{ \dots \mathbb{RP}^k \subset \mathbb{RP}^{k+1} \subset \dots \}$$

has a cell structure with $\mathbb{RP}^n = \bigsqcup_{0 \le k \le n} B^k$,

hence $C_k(\mathbb{RP}^n) = \mathbb{Z}$, $0 \le k \le n$,
$= 0$, otherwise.

The attaching maps $S^k \xrightarrow{\alpha_k} \mathbb{RP}^k$ are the quotient maps by $\{\pm 1\}$, and require some care: it turns out that

$$\begin{aligned} d_k = (-1)^k + 1 &= 2 \quad \text{if } k \text{ is even} \\ &= 0 \quad \text{if } k \text{ is odd,} \end{aligned}$$

$$0 \to \dots \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} 0.$$

$\boxed{6.2.2}$ Note that this argument also proves the <u>uniqueness</u> of (simplicial) homology at least on the class of spaces (homotopy equivalent to) cell complexes.

More precisely: Suppose H_* and h_* are two functors satisfying the ES axioms, *and* suppose there is a *natural transformation*

$$h_* \xrightarrow{\phi_*} H_*$$

of functors (ie, such that $\forall f: X \to Y$ $\exists$ commutative diagram

$$\begin{tikzcd} h_*(X) \arrow[r, "\phi_*(X)"] \arrow[d, "h_*(f)"'] & H_*(X) \arrow[d, "H_*(f)"] \\ h_*(Y) \arrow[r, "\phi_*(Y)"] & H_*(Y) \end{tikzcd}$$

)

such that $\phi_*(pt): h_*(pt) \to H_*(pt)$ is an isomorphism: then $\phi_*(X)$ is an isomorphism on any cell complex. [*Proof*: first verify that $\phi_*(S^n)$ is an isomorphism on spheres...]

$\boxed{6.2.3}$ The *five lemma* is an extremely useful fact from homological algebra, which provides an alternate proof of the uniqueness of homology (on the category of cell complexes) → $\boxed{\text{Kosinski Th 5.10 p 98}}$

Proposition: suppose

$$\begin{tikzcd} \arrow[r] & A_* \arrow[r] \arrow[d, "\alpha_*"'] & B_* \arrow[r] \arrow[d, "\beta_*"'] & C_* \arrow[r] \arrow[d, dashed, "\gamma_*"'] & A_{*-1} \arrow[r] \arrow[d, "\alpha_{*-1}"'] & B_{*-1} \arrow[r] \arrow[d, "\beta_{*-1}"'] & C_{*-1} \arrow[r, dash] \arrow[d, dashed, "\gamma_{*-}"] & {} \\ \arrow[r] & A'_* \arrow[r] & B'_* \arrow[r] & C'_* \arrow[r] & A'_{*-1} \arrow[r] & B_{*-1} \arrow[r] & C_{*-1} \arrow[r, dash] & {} \end{tikzcd}$$

is a commutative diagram with exact rows. Then (ie four out of five) if $\alpha_*, \beta_*, \alpha_{*-1}, \beta_{*-1}$ are isomorphisms, so is γ_*.

—— // ——

Application: If $Y = X \cup_\alpha B^n$ is formed by attaching a cell, and if $\phi_*: h_* \to H_*$ is a natural transformation of homology theories, then $\phi_*(X)$ iso $\Rightarrow$ $\phi_*(Y)$ iso:

$$\begin{tikzcd} \arrow[r] & h_*(X) \arrow[r] \arrow[d, "\cong"', "\phi_*(X)"] & h_*(Y) \arrow[r] \arrow[d, "\phi_*(Y)"] & h_*(Y,X) \cong h_*(S^n) \arrow[r] \arrow[d, "\cong"', "\phi_*(S^n)"] & \cdots \\ \arrow[r] & H_*(X) \arrow[r] & H_*(Y) \arrow[r] & H_*(Y,X) \cong H_*(S^n) \arrow[r] & {} \end{tikzcd}$$

(see above about iso on spheres

(Consequently ϕ_* is an iso on *any* cell cx (by induction on cells)

xiii)

$\boxed{6.2.4}$

Although it is technically out of reach at the moment (the proof involves some systematic study of homotopy groups, cf eg [Hatcher Ch4.1 p346]), it seems reasonable to mention here a theorem of JHC Whitehead:

Theorem Suppose $f:(X,*) \to (Y,*)$ is a map of pointed spaces (ie cell complexes!), such that

1) $\pi_1(X,*)$ and $\pi_1(Y,*)$ are abelian, and

2) $f_*: H_*(X, \mathbb{Z}) \to H_*(Y, \mathbb{Z})$ is an isomorphism. Then f is a homotopy equivalence.

Remark The restriction to abelian fundamental groups can be removed, by asking instead that f induce an isomorphism for all "twisted coefficients"; but that involves explaining what there are.

xiv)

The point of bringing this up here is that

1) the techniques involved in the proof of Whitehead's theorem are not so different from the proofs above (ie, by induction on cells), and

2) this theorem shows that singular homology (perhaps with a few bells and whistles) is enough to capture, at least to some degree, the homotopy type of a reasonable space (= cell complex).

Note that this theorem certainly does NOT say that spaces with (abstractly) isomorphic homology groups are homotopy equivalence: the isomorphism must come from some map between the spaces!

6.2.5 An important consequence of Whitehead's theorem is that the composition

$$X \mapsto |X[\bullet]|$$

(which assigns to a space of CW type, the geometric realization of its associated simplicial set) is a homotopy equivalence.

Following Dan Kan, this has led to an entirely combinatorial approach to homotopy theory, starting from the category Δ^{op}(Sets). It is technically very clean, particularly when questions about function spaces arise; but it requires some categorical sophistication; in particular, the homotopy category is defined by <u>declaring</u> a certain class of maps to be invertible . . .

6.3.1 With these results as background, we can return more honestly to the account of the homotopy category (HoT) of "reasonable" pointed spaces (introduced provisionally in §3.5 (p xv),) and interpret "reasonable" as meaning, "of the homotopy type of a cell complex".*

We can now summarize most of the preceding discussion as

1) The construction of a functor (singular chains, §5.2) from the category of (pointed) cell complexes and continuous maps, to the category of chain complexes, which <u>factors</u> <u>through</u> the category (HoT) of reasonable spaces and takes values in the category Hcx (§5.1 p vii).

* The subcategory of spaces 'of finite type' — ie, of the homotopy type of a cell complex with <u>finitely</u> <u>many</u> cells in <u>each</u> <u>dimension</u> — is particularly convenient

xvii)

of chain complexes and $\overset{\text{chain}}{\text{homotopy}}$ classes of chain homomorphisms :

$$\begin{tikzcd}
\text{(Reasonable Spaces)} \arrow[d] \arrow[r, "\text{singular homology}"] & (Ab_*) \\
\boxed{(HoT) \dashrightarrow (HC_*)} \arrow[r, "H_*"'] & (Ab_*)
\end{tikzcd}$$

It is at this factorization where, to be honest, 'the rubber meets the road': that is, most of the $\underline{\text{work}}$ in $\overset{\text{classical}}{\text{algebraic}}$ topology is done at this level, between the homotopy theory of spaces and the homotopy theory of algebraic (chain) complexes.

The object of this subsection is to push this factorization a little further, by noting that the $\underline{\text{suspension}}$ functor $X \mapsto S^1 \wedge X$ $\boxed{\S 3.6 \text{ p xxii}}$ on pointed spaces has an algebraic analog

xviii)

$$(C_*, \partial_*^C) \mapsto (C_{*-1}, \partial_{*-1}^C) := S^1(C_*, \partial_*^C), \quad (\S 5.1 \text{ p iv}) = (C_*, \partial_*) \otimes (\tilde{C}_*(S^1))$$

which can be identified with the tensor product of complexes, where $\tilde{C}_*(S^1)$

$$0 \to \overset{\tilde{C}_0}{0} \to \overset{\tilde{C}_1}{\mathbb{Z}} \to 0$$

is the 'reduced' chain complex of $S^1 = e_0 \cup B^1$.

$\underline{\text{Digression}}$: Graded vector spaces (or, more generally, R-modules) were defined in §3.2, along with their (graded) tensor products and (graded) modules of homomorphisms. In the discussion of chain complexes (§5.1) I neglected to note that the tensor product $(C_*, \partial_*^C) \otimes (D_*, \partial_*^D) := ((C \otimes D)_*, \partial_*^{C \otimes D})$ of two chain complexes is again a chain complex, with differential

$$\partial_k^{C \otimes D}(c \otimes d) = \partial_i^C c \otimes d + (-1)^{|c|} c \otimes \partial_j^D d, \quad (c \in C_i,\ d \in D_j,\ i+j=$$

and that $\mathrm{Hom}_*(C_*, D_*)$ is similarly a chain complex, with

$$\partial_i^{\mathrm{Hom}(C,D)}(\phi) = \partial_i^D \circ \phi + (-1)^i \phi \circ \partial_{i-1}^C ,$$

$\phi \in \mathrm{Hom}_i(C,D)$.

[*Exercise* Check that these are in fact differentials!]

With these definitions, we can lift the suspension isomorphism

$$\tilde{H}_i(S^1 \wedge X) = \tilde{H}_{i-1}(X)$$

of any homology theory to the assertion of a chain equivalence

$$\tilde{S}_*(S^1 \wedge X) \cong \tilde{S}_*(S^1) \otimes \tilde{S}_*(X)$$

of complexes.

Note, that tensoring with $\tilde{S}_*(S^1)$ — which amounts to shifting the grading by *one* — is

an automorphism of the category of chain complexes, with an inverse defined by shifting degrees in the opposite direction by one.

There is, however, no such *automorphism* of the homotopy category, because we neglected to invent a (-1)-dimensional sphere. Our next objective is to remedy this oversight!

Proposition If X, Y are finite cell complexes, then $\pi_0 \mathrm{Maps}_*(S^2 \wedge X, S^2 \wedge Y)$ is an abelian group. If Z is a third finite cell complex, the composition

$$\pi_0 \mathrm{Maps}_*(S^2 \wedge X, S^2 \wedge Y) \times \pi_0 \mathrm{Maps}_*(S^2 \wedge Y, S^2 \wedge Z) \to \pi_0 \mathrm{Maps}(S^2 \wedge X, S^2 \wedge Y)$$

is a groups homomorphism.

xxi)

<u>Proof</u>: of §3.5:

$$\pi_0 \mathrm{Maps}_*(S^2 \wedge X, S^2 \wedge Y) =$$

$$\pi_0 \mathrm{Maps}_*(S^2, \mathrm{Maps}(X, S^3 \wedge Y)) =$$

$$\pi_2 \mathrm{Maps}(X, S^2 \wedge Y)$$

is an abelian group. Moreover

$$\pi_0 \mathrm{Maps}_*(S^1 \wedge X, S^1 \wedge Y) = \pi_1 \mathrm{Maps}_*(X, S^1 \wedge Y)$$

is a group, and

$$\pi_0 \mathrm{Maps}(S^1 \wedge X, S^1 \wedge Y) \times \pi_0 \mathrm{Maps}(S^1 \wedge Y, S^1 \wedge Z) \to \pi_0 \mathrm{Maps}(S^1 \wedge X, S^1 \wedge Z)$$

is a group homomorphism.

<u>Definition</u> The suspension endofunctor $X \mapsto S^1 \wedge X$ defines a homomorphism

$$\pi_0 \mathrm{Maps}_*(S^n \wedge X, S^n \wedge Y) \to \pi_0 \mathrm{Maps}(S^{n+1} \wedge X, S^{n+1} \wedge Y)$$

abelian groups, when $n \geq 2$. The

xxii)

direct limit

$$\varinjlim \{ \pi_0 \mathrm{Maps}(S^n \wedge X, S^n \wedge Y) \mid \text{suspension} \}$$

$$:= \{X, Y\}_0$$

of the resulting directed system of abelian groups is the group of <u>stable</u> <u>homotopy</u> <u>classes</u> of maps from X to Y. [More generally,

$$\{X, Y\}_k := \{S^k \wedge X, Y\}_0$$

is defined for all $k \in \mathbb{Z}$ (but $\doteq 0$ for $k < 0$).

The resulting maps

$$\{X, Y\}_k \times \{Y, Z\}_\ell \longrightarrow \{X, Z\}_{k+\ell}$$

are <u>associative</u>, thus defining the <u>stable</u> <u>homotopy</u> <u>category</u> (SHoT) of <u>finite</u> (pointed) (graded) cell <u>complexes</u>. It is a <u>additive</u> category.

<u>Proposition</u> The functor

$$X \mapsto S_*(X) : (\mathrm{Hot}) \to (\mathrm{HCx})$$

factors through the additive category (SHoT), ie such that the maps

$$[f] \mapsto S_*(f) : \{X, Y\} \longrightarrow \mathrm{Hom}_{\mathrm{HCx}}(S_*(X), S_*(Y))$$

are homomorphisms of abelian groups.

<u>Proof</u>: If $f, g : S^2 \wedge X \to Y$ are maps of spaces, the pinch map on S^2 defines a composition

$$S^2 \wedge X \longrightarrow (S^2 \vee S^2) \wedge X \xrightarrow{f \vee g} Y \; ;$$

but $S_*(S^2 \wedge X \vee S^2 \wedge Y) \cong S_*(S^2 \wedge X) \oplus S_*(S^2 \wedge Y)$.

—//—

It follows immediately that this extends to the associated <u>graded</u> category, ie that

$$\{X, Y\}_k \longrightarrow \mathrm{Hom}^{\mathrm{HCx}}_k(S_*(X), S_*(Y))$$

is a map of abelian groups. ▭

$\boxed{6.3.2}$ An arguably better (ie more natural) way to approach these issues is to introduce the category of (finite, naive) <u>spectra</u>, having families of maps (of finite cellular spaces)

$$\underline{\underline{X}} := \{S^n \wedge X_m \xrightarrow{f^n_m} X_{n+m}\} \quad /\!/ \quad \gg 0$$

such that

$$S^k(f^n_m) = f^{k+n}_m : S^k(S^n \wedge X_m) \to S^k(X_{n+m}) \to X_{n+}$$

as objects, with commuting diagrams

$$\begin{tikzcd}
S^n \wedge X_m \arrow[r, "f^n_m"] \arrow[d, "S^n(\phi_m)"'] & X_{n+m} \arrow[d, "\phi_{n+m}"] \\
S^n \wedge Y_m \arrow[r, "g^n_m"] & Y_{n+m}
\end{tikzcd}$$

of maps as morphisms. The associated homotopy category has

$$\lim_{\longleftarrow g} \; \lim_{\xrightarrow{f}} \; \text{to Maps}(X_n, Y_m) := \{X, Y\}^S_0$$

[this requires an important stability theorem of Freudenthal

as morphisms; by arguments similar to the constructions above these can be shown to be (finitely-generated!) abelian groups. In a natural sense the homotopy category of spectra can be constructed from the stable homotopy of finite complexes obtained by adjoining 'negative-dimensional' spheres

$$\underline{\underline{S}}^{-k} = \{ S^n \wedge S^{-k} := S^{n-k} \xrightarrow{\cong} S^{n-k} \}, n \gg 0)$$

Much as I would like, I will leave this topic aside, only remarking that these ideas can (in various, ultimately equivalent ways) be extended/ to define a symmetric (ie internal Homs & a smash product) monoidal category of spectra. One elegant technical variant (the cat of symmetric spectra) requires the spaces X_n to carry a Σ_n-action, such that the

maps

$$S^n \wedge X_m \longrightarrow X_{n+m}$$

are Σ_{n+m}-equivariant, with S^n regar as the sphere $\mathbb{R}^n_+$ associated to the regular representation of Σ_n on $\mathbb{R}^n$. [This allows the construction of a good! smash product structure on the category.]

As above, the singular chain complex defines an additive functor from "the" homotopy category of spectra to the homotopy categ (bounded neither above nor below) chain complexes of R-modules. More significantly, however, a theorem of EH Brown asserts that any extraordinary homology theory (ie, a xxvii)

$$E_*(-)$$

homotopy functor / from reasonable spaces to graded abelian groups, which satisfies all the ES axioms *except* for the dimension axiom ($E_*(pt) = 0$ if $* \neq 0$), is defined by a spectrum $\underline{\underline{E}}$, ie

$$E_*(X) \cong \{S^*, X \wedge \underline{\underline{E}}\}.$$

Note that homology theories which 'lift' to the category of chain complexes (in a certain natural sense relating to their behavior on 'cofibration' sequences of the form $A \to X \to X \cup CA$ (§3.7 p xxiii)) are *necessarily* naturally isomorphic to some form of singular homology.

xxviii)

Remark The infinite symmetric product

$$SP^\infty(X) = \bigvee_{n \geq 0} (X^{\wedge n})/\Sigma_n$$

of a pointed space X can be defined as the free abelian topological monoid generated by a pointed space X, with the base-point as identity element. [This is closely related to the notion of 'divisors' in algebraic geometry.] A deep theorem of Dold and Thom asserts that the homotopy groups

$$\pi_* SP^\infty(X) \cong H_*(X, \mathbb{Z})$$

of this monoid are naturally isomorphic to the homology groups of X. This is closely related to the free *simplicial abelian group* $\mathbb{Z}[X[\cdot]]$ defined by the simplicial set $X[\cdot]$.

§7 Cohomology

7.1 basics

Notational convention: (cf. §3.2 p xi) graded abelian groups $A_* = \bigoplus_{k \in \mathbb{Z}} A_k \in \mathrm{Ab}_*$ form a category, with graded abelian groups

$$\mathrm{Hom}_*(A_*, B_*) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(A_i, B_{i+k})$$

as homomorphisms between them. It will be convenient to write $A_{-*} := \overline{A}^{*}$ sometimes.

Similarly, for modules over a ring R, eg R=Z

Definition / Claim If R is a commutative ring, and $M \in R\text{-Mod}$, there is a contravariant functor from (reasonable) pairs of spaces to graded R-modules, [called cohomology $H^*(X, M)$ with coeffs in M] such that (fill in the blank). In particular,

for a pair $A \subset X$ ∃ long exact sequence

$$\to H^{i-1}(A) \xrightarrow{\delta^{i-1}} H^i(X, A) \to H^i(X) \to H^i(A) \to \cdots$$

(cf notation)

(note change/ from ∂).

Construction

Recall (§5.2 p xiv) that the group $S_n(X, \mathbb{Z})$ of singular chains, with coefficients in $\mathbb{Z}$, of a space X is the free abelian group $(\bigoplus_{X[n]} \mathbb{Z})$ generated by (oriented) singular simplices in X.

The group

$$S^n(X, \mathbb{Z}) = \left(\prod_{X[n]} \mathbb{Z}\right)$$

of singular n-dimensional cochains on X is the set of functions from the set of oriented simplices in X, to $\mathbb{Z}$.

Equivalently, (§1 p xx): $\mathrm{Hom}(\bigoplus_S \mathbb{Z}, \mathbb{Z}) = \prod^S \mathbb{Z}$ etc.

$$S^n(X, \mathbb{Z}) = \mathrm{Hom}(S_n(X, \mathbb{Z}), \mathbb{Z})$$

(= $\mathrm{Hom}_{\bar{n}}(S_*(X, \mathbb{Z}), \underline{\mathbb{Z}})$

where $\underline{\mathbb{Z}}$ is a chain cx with only $\mathbb{Z}$ in deg 0).

Corollary $\delta^n = \mathrm{Hom}(\partial_{n+1}, \mathbb{Z})$ defines the *cochain complex*

$$\cdots \to S^{n-1}(X,\mathbb{Z}) \xrightarrow{\delta^{n-1}} S^n(X,\mathbb{Z}) \xrightarrow{\delta^n} S^{n+1}(X,\mathbb{Z}) \to \cdots$$

(satisfying $\delta^n \circ \delta^{n-1} = 0$). Its homology is the *cohomology* of X.

Recalling that the complex $S_*(X,A)$ of relative chains is defined as the quotient complex

$$0 \to S_*(A) \to S_*(X) \to S_*(X,A) \to 0,$$

the complex $S^*(X,A)$ is the kernel

$$0 \to S^*(X,A) \to S^*(X) \to S^*(A) \to 0$$

of the dual complex; alternately, $S^*(X,A)$ can be identified by the complex of functions from $X[n]$ to $\mathbb{Z}$, which *vanish* on the subset $A[n]$.

Remark Since the category of abelian groups is close to self dual, this construction isn't much of a surprise. Verification of the (contravariant analogs of) the ES axioms are immediate. So, why bother?

* *Further* *properties* of the coh. functor:

(If R is a commutative ring, eg $\mathbb{Z}$ ...) then

•) $H^*(X,R)$ is a *graded-commutative* ring:

$x \in H^p(X), y \in H^q(X) \mapsto x \smile y \in H^{p+q}(X)$,

$x \smile y = (-1)^{pq} y \smile x$. This product is natural in that if $f: X \to Y$ and $u, v \in H^*(Y)$ then

$$f^*(u \smile v) = f^*(u) \smile f^*(v).$$

•) In fact $H^*(X,A)$ is an $H^*(X)$-module, compatibly with $H^*(X,A) \to H^*(X)$;

•) $H_*(X,R)$ is a graded $H^*(X,R)$-module: if $x, y \in H^p, H^q$ and $z \in H_n$ then $\exists\ x \cap z \in H_{n-p}$, $(x \cup y) \cap z = x \cap (y \cap z)$.

d) This product is natural in that if f is above then the diagram

$$\begin{tikzcd}
H^*(Y) \otimes H_*(X) \arrow[r, "1 \otimes f_*"] \arrow[d, "f^* \otimes 1_X"] & H^*(Y) \otimes H_*(Y) \arrow[dd, "\cap Y"] \\
H^*(X) \otimes H_*(X) \arrow[d, "\cap X"] & \\
H_*(X) \arrow[r, "f_*"] & H_*(Y)
\end{tikzcd}$$

ie $f_*(f^*y \cap x) = y \cap f_* x$, $x \in H_* X$, $y \in H^* Y$.

Construction The idea, of course, is to define a homomorphism

$$\cup : S^*(X) \otimes S^*(X) \longrightarrow S^*(X)$$

of chain complexes. But the boundary operator on the left will be, by definition

by definition be

$$\delta(c \otimes c') = \delta c \otimes c' + (-1)^{|c|} c \otimes \delta c' ,$$

(with $(\delta c)(\sigma) = c(\partial \sigma)$). If this holds, and c, c' are cocycles (ie $\delta c, \delta c' = 0$), then their product $c \otimes c'$ will be a cocycle; and if we perturb c to $c + \delta f$ then

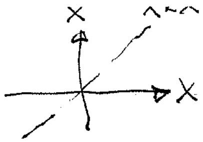
$$\begin{aligned} (c + \delta f) \otimes c' &= c \otimes c' + \delta f \otimes c' \\ &= c \otimes c' + \delta(f \otimes c') \\ &\qquad (\text{since } f \otimes \delta c' = 0) \end{aligned}$$

will represent the same class in $H^*(X)$.

In fact it is more natural to construct a chain homomorphism

$$S^*(X) \otimes S^*(Y) \longrightarrow S^*(X \times Y).$$

In the special case when $X = Y$, we can then compose this with the homomorphism vii).



$$\Delta^* : S^*(X \times X) \longrightarrow S^*(X)$$

induced (recall S^* is <u>contravariant</u>) by the diagonal map $x \mapsto (x,x) : X \to X \times X$.
[Similarly for $S^*(X,A)$, use $X \to X \times X \to X \times (X/A)$.]

The argument is based on a principle pointed out to me by Mike Hopkins, ie that operations on simplicial chains (and cochains) are naturally parameterized by maps in the ~~category~~ category of simplicial complexes: in particular, there are standard <u>front</u> and <u>back</u> face maps

$$\lambda_i : \Delta^i \to \Delta^n \ , \ \rho_i : \Delta^i \to \Delta^n$$

defined by $\lambda_i(t_0, \ , t_i) = (t_0, \ldots, t_i, 0, \ldots, 0)$
$\rho_i(t_0, \ldots, t_i) = (0, \ldots, 0, t_0, \ldots t_i)$

[<u>Rotman</u> Ch 12 p391].

viii)

If $c \in S^n(X)$, $c' \in S^m(Y)$ then we can define

$$c \cup c' \in S^{n+m}(X \times Y)$$

as the function which sends $\sigma : \Delta^{n+m} \to X \times Y$ to $c(p_X \circ \sigma \circ \lambda_n) \cdot c'(p_Y \circ \sigma \circ \rho_m)$, with

$$\Delta^n \xrightarrow{\lambda_n} \Delta^{n+m} \xrightarrow{\sigma} X \times Y \xrightarrow{p_X} X$$
$$\Delta^n \xrightarrow{\rho_m} \Delta^{n+m} \xrightarrow{\sigma} X \times Y \xrightarrow{p_Y} Y$$

(p_X, p_Y are the projections to the appropriate factor).

The verification that δ acts as a derivation with respect to this product is in Rotman, (and should by now look familiar) p394–395; it is straightforward and I won't repeat it here.

<u>Note</u> that the construction above is <u>dual</u> to the <u>Alexander Whitney</u> chain homomorphism

ix)

$\alpha : S_*(X\times Y) \longrightarrow S_*(X)\otimes S_*(Y)$,

$\alpha(\sigma_n) = \sum_{p+q=n} (p_X\sigma\lambda_p)\otimes(p_Y\sigma\rho_q)$

<u>Proposition</u> The resulting product on $H^*(X)$ is (anti)-commutative: if $x\in H^p$, $y\in H^q$, then $x\cup y = (-1)^{pq}\, y\cup x$.

<u>Proof</u>, following [<u>Hatcher</u> 3.14 p 215]: If $\sigma : [v_0,\dots, v_n]\longrightarrow X \in S_n(X)$, let

$\bar\sigma : [v_n,\dots,v_0]\longrightarrow X$ be defined by reversing the ordering of the simplices (cf §5.2 p xiv); this is an element of sgn $\overset{\varepsilon_n =}{(-1)^{n(n+1)/2}}$ in Σ_n, so let $\rho_n : S_n(X)\longrightarrow S_n(X)$ be defined by

$\rho_n(\sigma) = (-1)^{n(n+1)/2}\,\bar\sigma$.

<u>Claim</u> ρ_n is a chain map, chain homotopic to

x)

the identity.

<u>Proof</u> 1) That the asserted proposition follows from this claim: we have

$(\rho^*c\cup\rho^*c')(\sigma) = c(\varepsilon_n\sigma|[v_n,\dots,v_0])\cdot c'(\varepsilon_m\sigma|[v_{n+m},\dots v_n])$

while

$\rho^*(c\cup c')(\sigma) = \varepsilon_{n+m}\, c'(\sigma|[v_{n+m},\dots,v_n])\cdot c(\sigma|[v_n,\dots,v_0]))$,

and $\varepsilon_{n+m} = (-1)^{nm}\,\varepsilon_n\cdot\varepsilon_m$ $\big(n(n+1)+m(m+1)+2mn = = (n+m)(n+m+1$

2) That ρ is in fact a chain map:

$\partial\rho(\sigma) = \varepsilon_n\sum(-1)^i\,\sigma|[v_n,\dots,\hat v_{n-i},\dots,v_0]$, while

$\rho\partial(\sigma) = \rho(\sum(-1)^i\sigma|[v_0,\dots,\hat v_i,\dots,v_n])$

$= \varepsilon_{n-1}\sum(-1)^{n-i}\sigma|[v_n,\dots,\hat v_{n-i},\dots,v_0]$;

but $\varepsilon_{n-1}(-1)^n = (-1)^{n(n-1)/2+n} = (-1)^{n(n+1)/2} = \varepsilon_n$

3) That ρ is chain homotopic to the identity

the identity:

We have to provide the chain homotopies

$$P_n : S_n(X) \to S_{n+1}(X) ;$$

let $\pi : \Delta^n \times I \to \Delta^n$ be the projection, and set

$$P_n(\sigma) = \sum (-1)^i \varepsilon_{n-i} (\sigma \circ \pi) | [v_0, \dots, v_i ; w_n, \dots, w_i]$$

(following the notation for the proof of the homotopy axiom, §5.2 p xv). Following Hatcher, we omit the $(\sigma \circ \pi)$ factors and check the identity

$$\partial P + P \partial = \rho - 1$$ as follows:

$$\partial P = \sum_{j \le i} (-1)^i (-1)^j \varepsilon_{n-i} [v_0, \dots, \hat{v}_j, \dots, v_i, w_n, \dots, w_i]$$
$$+$$
$$\sum_{j \ge i} (-1)^i (-1)^{i+n+1-j} \varepsilon_{n-i} [v_0, \dots, v_i ; w_n, \dots, \hat{w}_j, \dots, w_i]$$

the $i = j$ terms give

$$\varepsilon_n [w_n, \dots, w_0] + \sum_{i > 0} \varepsilon_{n-i} [v_0, \dots, v_{i-1}, w_n, \dots, w_i]$$

$$+ \sum_{i < n} (-1)^{n+i+1} \varepsilon_{n-i} [v_0, \dots, v_i, w_n, \dots, w_{i+1}] \;\&$$
$$- [v_0, \dots, v_n] .$$

The two summations cancel since replacing i by $i-1$ in the second sum produces a sign $(-1)^{n+1} \varepsilon_{n-i+1} = -\varepsilon_{n-i}$. The remaining terms represent $\rho(\sigma) - \sigma$, so it suffices to check that the $i \ne j$ terms yield $-P\partial$; but

$$P\partial = \sum_{i < j} (-1)^i (-1)^j \varepsilon_{n-i-1} [v_0, \dots, v_i, w_n, \dots, \hat{w}_j, \dots, w$$
$$+ \sum_{i > j} (-1)^{i-1} (-1)^j \varepsilon_{n-i} [v_0, \dots, \hat{v}_j, \dots, v_i ; w_n, \dots, w_i] ,$$

and since $\varepsilon_{n-i} = (-1)^{n+1} \varepsilon_{n-i-1}$ we are through.

$\boxed{7.2 \text{ Some examples}}$

mostly (i.e. entirely) without proofs:

•) $H^*(S^n, \mathbb{Z}) \cong \mathbb{Z}[e_n]/(e_n^2)$.

Note that the relation is forced by anticommutativity when n is odd.

x(iii))

•) $H^*(\mathbb{C}P^n, \mathbb{Z}) \cong \mathbb{Z}[t]/(t^{n+1})$

In other words, $H^i(\mathbb{C}P^n, \mathbb{Z}) = 0$ i odd
$= \mathbb{Z}$ i even, $0 \leq i \leq 2n$,
in accord with the cell structure; with all products nontrivial.

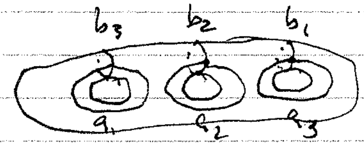
Similarly, $H^*(\mathbb{C}P^\infty, \mathbb{Z}) = \mathbb{Z}[t]$ is the free polynomial algebra on a generator of degree two

•) If F_g is a closed orientable surface of genus g, then

$$H^*(F_g, \mathbb{Z}) = \begin{array}{ll} \mathbb{Z} & \deg 2 \\ \mathbb{Z}^{2g} & \deg 1 \\ \mathbb{Z} & \deg 0 \end{array}$$

with generators $a_1, \ldots, a_g; b_1, \ldots, b_g \in H^1(F_g, \mathbb{Z})$,

such that $a_j \smile a_k = 0$
$b_i \smile b_k = 0$
$a_i \smile b_k = \delta_{ik} \cdot$ (generator of $H^2(F_g, \mathbb{Z})$



This is intended to suggest thinking of the product as a kind of intersection.

x(iv))

•) a $\overbrace{\text{polynomial}}^{\text{(generic)}}$ $F(x,y) \in \mathbb{C}[x,y]$ of degree d defines a (smooth) surface $Z_F = \{[X:Y:Z] \mid Z^d F(\frac{X}{Z}, \frac{Y}{Z}) = 0\} \subset \mathbb{C}P^2$ of genus $g = \frac{1}{2}(d-1)(d-2)$, defining a ring homomorphism

$$\begin{tikzcd}[row sep=small] & H^*(\mathbb{C}P^2; \mathbb{Z}) & H^*(Z_F; \mathbb{Z}) \\ \dim 4 \cong & \mathbb{Z} \ni t^2 \arrow[r] & 0 \\ & 0 & 0 \\ \dim 2 \cong & \mathbb{Z} \ni t \arrow[r, mapsto] & d \cdot 1 \in \mathbb{Z} \\ & 0 \arrow[r] & \mathbb{Z}^{2g} \\ \dim 0 \cong & \mathbb{Z} \ni 1 = t^0 \arrow[r, "\cong"] & \mathbb{Z} \end{tikzcd}$$

•) $\exists$ quaternionic projective space
$\mathbb{H}P^n = (\mathbb{H}^{n+1} - 0)/(\mathbb{H} - 0)$, with $\left\{\begin{array}{l}\text{one cell} \\ \text{in each} \\ \dim \equiv 0 \\ \bmod 4\end{array}\right.$

$H^*(\mathbb{H}P^n, \mathbb{Z}) = \mathbb{Z}[z]/(z^{n+1})$, where $\underline{|z| = 4}$

Lack of associativity prevents the construction of a projective space over the Cayley numbers, xv)

but there does exist a Cayley projective plane $\mathbb{O}P^2$ with

$$H^*(\mathbb{O}P^2;\mathbb{Z}) = \begin{array}{ll} \mathbb{Z}\ni\xi^2 & \dim 16 \\ \mathbb{Z}\ni\xi & \dim 8 \\ \mathbb{Z}\ni\xi^0 & \dim 0 . \end{array}$$

Why this is so is connected to the Hopf invariant one problem...

•) $H^*(\Omega S^{2n-1};\mathbb{Z}) = \mathbb{Z}[b_k \mid k\geq 1]$

modulo relations $b_k\cdot b_\ell = (k,\ell)\, b_{k+\ell}$

with $(k,\ell) = \frac{(k+\ell)!}{k!\,\ell!}$:

in other words, b_k behaves like the 'divided power' $\rightsquigarrow \beta^k/k!$ of an element β of degree n,

$$\frac{\beta^k}{k!}\cdot\frac{\beta^\ell}{\ell!} = \frac{(k+\ell)!}{k!\,\ell!}\,\frac{\beta^{k+\ell}}{(k+\ell)!} .$$

xvi)

•) The example above, continued: when $n=2$,

$$S^{2n-1} = S^3 = SU(2) = \text{unit sphere in the quaternions } \mathbb{H}\cong\mathbb{R}^4$$

is a topological group, which implies that the based loopspace ΩS^3 is a group as well. [Why?] This group can be shown to have a faithful projective Hilbert-space representation (important for ex in quantum field theory), which leads to the existence of map (in fact an embedding)

$$\Omega S^3 \longrightarrow \mathbb{CP}^\infty .$$

On cohomology we get a map

divided powers on β

$$H^*(\mathbb{CP}^\infty,\mathbb{Z}) \cong \mathbb{Z}[t] \longrightarrow \mathbb{Z}[b_k|k\geq1] \cong H^*(\Omega S^3,\mathbb{Z}$$

which sends t^k to $k!\,b_k$ ($=\beta^k$, if β had existed).

Problem what does $H^*(\mathbb{CP}^\infty/\mathbb{CP}^n,\mathbb{Z})$ look like?

xvii)

$\boxed{\text{7.3 Cap products}}$

As sketched in §7.1, to construct the cap product $\underset{X}{\cap}: H^p(X) \otimes H_q(X) \to H_{q-p}(X)$ it suffices to construct a chain homomorphism

$$S^*(X) \otimes S_*(X) \longrightarrow S_*(X).$$

If $c \in S^p(X)$ and $\sigma : \Delta^q \to X$ then

$$\boxed{c \cap \sigma \ := \ c(\sigma|[v_0, \ v_p]) \cdot \sigma|[v_p, \dots v_q]}$$

is a $(q-p)$-chain in X, which satisfies

$$\partial(c \cap \sigma) = \delta c \cap \sigma + (-1)^p c \cap \partial\sigma .$$

check, following [<u>Hatcher</u> §3.3 p 240]: The sum on the right, above, can be written

✓ which sums to $\partial(c(\sigma))$!

$$\sum_{i=0}^{i=p+1} (-1)^i c(\sigma|[v_0, \dots, \hat{v}_i, \dots, v_{p+1}]) \cdot \sigma|[v_{p+1}, \dots, v_q]$$

$$+$$

$$(-1)^p \Big[\sum_{i=0}^{i=l} (-1)^i c(\sigma|[v_0, \dots, \hat{v}_i, \dots, v_{l+1}]) \cdot \sigma|[v_{l+1}, \dots, v_q]$$

$$+$$

$$\sum_{i=l+1}^{l=q} c(\sigma|[v_0, \dots, v_l]) \cdot \sigma|[v_l, \dots, \hat{v}_i, \dots, v_k] \Big]$$

xviii)

<u>Corollary</u> It is useful to know, and straightforward to prove, that ~~&~~ with coefficients in a <u>field</u> $R = F$, the cap product pairing

$$H^p(X, F) \otimes_F H_p(X, F) \longrightarrow F$$

is <u>perfect</u>, eg that

dual vector space

$$H^p(X, F) \longrightarrow \mathrm{Hom}(H_p(X, F), F) = H_p(X, F)^\vee$$

is an <u>isomorphism</u>. [Note by the way that this is consistent with our conventions relating upper and lower indices — <u>for objects graded by $\mathbb{N}$, not $\mathbb{Z}$</u>!

By duality we can thus regard $f_* : H_*(X, F) \to H_*(Y, F)$ as an element of $H^*(X, F) \otimes_F H_*(Y, F)$

Note by the way that $f : X \to Y$ thus induces

$$f^* : H^*(Y, F) \to H^*(X, F) \cong \mathrm{Hom}(H_*(X, F), F)$$

$$\in \mathrm{Hom}_F(H^*(Y, F), \mathrm{Hom}(H_*(X, F), F))$$

$$\cong \mathrm{Hom}_F(H^*(Y, F) \otimes_F H_*(X, F), F)$$

Cap products are a key tool in the proof of the Poincaré duality theorem, and we will need a slight sharpening of their construction.

Claim We have well behaved relative cap products

$$H^p(X,A) \otimes H_q(X,A) \longrightarrow H_{q-p}(X)$$

and

$$H^p(X) \otimes H_q(X,A) \longrightarrow H_{q-p}(X,A)$$

Proof: If $c \in S^p(X,A)$ is a cochain in X which vanishes on simplices in A, and $\sigma \in S_q(X)$ actually lives in the submodule $S_q(A)$, then

$$c \frown \sigma = \underset{\overset{\shortparallel}{0}}{c(\sigma|[v_0,..,v_p])} \cdot \sigma|[v_p,..,v_q] = 0;$$

hence $c \frown -$ is well-defined as a homomorphism from $S_q(X,A)$ to $S_{q-p}(X)$. The second assertion is similar.

⑦.4 Foreshadowing the Künneth theorem

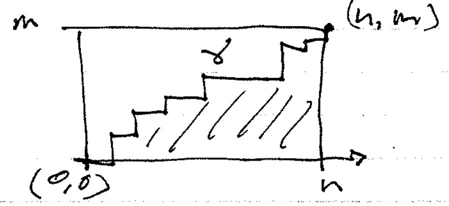
The construction of cap product using the Alexander Whitney map in §7.1 suggests the possibility of calculating the (co)homology of a product in terms of the (co)homology of its factors. The Künneth theorem in fact asserts that, with field coefficients,

$$H_*(X \times Y, F) \cong H_*(X,F) \otimes_F H_*(X,F)$$

(and similarly for cohomology).

A more general version of this, valid for any coefficients, is a consequence of the Eilenberg-Zilber theorem, which asserts (in the language of §6.3 p xvii) that the

An ascending path 'γ'



from (0,0) to (n, m) on the associated integral grid is the graph of a function

$$\gamma : [n+m] \longrightarrow [n] \times [m]$$

which thus defines a simplex in $\Delta^n \times \Delta^m$. If $|\gamma|$ is the number of squares below the graph it can be shown [Hatcher, §3.8 p 278] that if $[\sigma : \Delta^n \to X] \in S_n(X)$, $[\sigma' : \Delta^m \to Y] \in S_m(Y)$, then

$$\beta(\sigma \otimes \sigma') = \sum (-1)^{|\gamma|} [(\sigma \times \sigma') \circ \gamma : \Delta^{n+m} \to X \times Y]$$

defines a chain homomorphism

$$S_*(X) \otimes S_*(Y) \to S_*(X \times Y),$$

simplicial complex

$$S_* : (HoT) \longrightarrow (HCx)$$

viewed as a functor between the homotopy category of spaces and the homotopy category of chain complexes, preserves products (ie is 'monoidal'): there is a natural chain homotopy equivalence

$$S_*(X) \otimes S_*(Y) \underset{\alpha}{\overset{\beta}{\rightleftarrows}} S_*(X \times Y),$$

where α is the Alexander Whitney map, and β is the Eilenberg-Zilbert map, constructed by astutely decomposing the product $\Delta^n \times \Delta^m$ as a union of Δ^{n+m}'s:

Recall (§ 4.4 p xviii) that the vertices of the simplicial complex $\Delta^n \times \Delta^m$ are indexed by pairs $(i,j) \in [0,n] \times [0,m]$, given the dictionary order homotopy - inverse to α.

The most efficient proof of this, and many other results, uses the method of acyclic models [Rotman Ch 9 p237], which I leave for next semester. The point is that this topic is a gateway into the study of homological algebra, and which provides natural answers, such as the relations between (co)homology groups for different coefficients, which I have systematically avoided.

Example In §6.21 (px) above, a cell decomposition of $\mathbb{RP}^n$ can be used to show that

$\tilde{H}_*(\mathbb{RP}^n, \mathbb{Z}) = 0$ if i is even,

$\tilde{H}_*(\mathbb{RP}^n, \mathbb{Z}) = \mathbb{Z}_2$ if i is odd and $< n$

and $= 0$ otherwise; whereas

$H^*(\mathbb{RP}^n, \mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^{n+1})$

is a well-behaved polynomial algebra.

This is explained by the theory of derived functors (of ⊗ and Hom, in particular), which was a part of algebra which developed out of these questions in the 1950's (and which mystified the geometers of earlier generations).

§ 8 Poincaré duality

8.1 intro Manifolds are locally Euclidean spaces, and form the geometric foundations of physics; they are thus of profound scientific interest. The methods of algebraic topology presented here, however, are homotopy-invariant; and being locally Euclidean is not a homotopy-invariant property. Poincaré duality is important because it characterizes such local properties in global terms.

For example, if M is a compact, orientable, n-dimensional manifold without boundary, then

$$H^*(M, \mathbb{Z}) \cong \mathbb{Z} \text{ if } * = 0$$
$$= 0 \text{ if } * > n,$$

and the cup product pairing

(as a consequence of P. duality)

$$H^i(M, \mathbb{Z}) \otimes H^{n-i}(M, \mathbb{Z}) \to \mathbb{Z}$$

is nondegenerate modulo torsion; in particular, if n = 4k (eg our own dear Universe?), the pairing on the middle-dimensional cohomology groups defines a unimodular symmetric quadratic form (ie, represented by an integral matrix with det = ±1)

$$q_M : H^{2k}(M, \mathbb{Z}) \to \mathbb{Z},$$

a class of objects whose great interest was first noticed by Gauss.

For example, a generic quartic (exalgebraic) "K3" surface in CP(3), eg defined by an equation of the form

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 + \alpha z_0 z_1 z_2 z_3 = 0$$

defines a smooth compact four-dimensional manifold with Euler characteristic $\chi = 24$, and (co)homology groups

H^4	$\mathbb{Z}$
	0
H^2	$\mathbb{Z}^{22}$
	0
H^0	$\mathbb{Z}$

, ~~with the~~ its quadratic form on H^2 being the sum of three copies of the 'hyperbolic' form $x, y \mapsto x \cdot y$ with two copies of the quadratic form E_8 (defined by the weight lattice (seriously mysterious) of the Lie group of the same name).

Pursuing these questions further identifies manifolds as spaces whose cohomology forms a *Frobenius* algebra, manifesting a kind of homotopy-theoretic self-duality, of which Poincaré's result is just the tip of the iceberg.

The orientation class

8.2 There is in fact a whole family of Poincaré duality theorems, under various hypotheses; but all their proofs involve a kind of local-to-global induction, and require consideration of manifolds which may be neither compact nor without boundary. To see how these results fit together, recall that the boundary ∂M of a (compact, orientable) n-dimensional manifold has a (compact, orientable] $(n-1)$-dimensional boundary ∂M. We then have a

THEOREM $\exists$ comm. diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & H^{*-1}(\partial M) \arrow[r, "\delta"] \arrow[d, "\cong"] & H^*(M, \partial M) \arrow[r] \arrow[d, "\cong"] & H^*(M) \arrow[r] \arrow[d, "\cong"] & H^*(\partial \\
\cdots \arrow[r] & H_{n-*}(\partial M) \arrow[r] & H_{n-*}(M) \arrow[r, "\partial"] & H_{n-*}(M, \partial M) \arrow[r] & {}
\end{tikzcd}$$

and vertical *isomorphisms*. (cf eg *Hatcher* §3 p260)

The <u>rows</u> are the exact homology and cohomology sequences of the pair $(M,\partial M)$ [note that they aren't quite aligned in the most obvious way] and the vertical isomorphisms are defined by cap products with certain 'fundamental classes' $[M,\partial M]\in H_n(M,\partial M)$, $[\partial M]\in H_{n-1}(\partial M)$.

In fact such a theorem holds more generally, for (connected) noncompact manifolds with 'collared' boundaries, provided we replace ordinary cohomology with its "compactly supported" analog H_c^* (defined in terms of one-point compactification).

In particular, if $\partial M=\emptyset$ the assertion above simplifies to an isomorphism

$$x \mapsto x\cap[M] \ : \ H^*(M) \xrightarrow{\cong} H_{n-*}(M)$$

defined by an orientation class $[M]$ generating $H_n(M,\mathbb{Z})$ as stated (but not proven) in §3.8.1. The argument for this assertion is a kind of toy model for the proof of the duality theorem itself, and I have postponed its completion till now, as a warmup exercise.

following [<u>Greenberg</u> 22.24 p120, 26.6 p164]

<u>Recall</u> that (for A closed in M) the proposition defines a homomorphism from $H_n(M,M-A)$ to the module of sections of the orientation sheaf over A (which has 'stalk' $H_n(M,M-x)$ above x) as the map induced by

$$(M,M-A)\to(M,M-x)\,,\qquad x\in A.$$

The <u>claim</u> is that this identifies $H_n(M,M-A)$

with the module of compactly supported sections of the orientation sheaf.

Lemma Suppose the assertion above is true for closed sets A, B, and $A \cap B$ in M; then it is true for $A \cup B$.

This follows from a relative Mayer-Vietoris sequence

$$\begin{tikzcd}
0 \arrow[r] & H_n(X, X - A\cup B) \arrow[r] \arrow[d, "j_{A\cup B}"] & H_n(X, X-A) \oplus H_n(X, X-B) \arrow[r] \arrow[d, "j_A \oplus j_B"] & H_n(X, X-(A B)) \arrow[d, "j_{A\cap B}"'] \\
0 \arrow[r] & M_c^{or}(A\cup B) \arrow[r] & M_c^{\alpha}(A) \oplus M_c^{\alpha}(B) \arrow[r] & M_c^{\alpha}(A \cap B)
\end{tikzcd}$$

and the sheaf property for the orientation sheaf: the top row has 0 on the left by the induction (§ 3.8 p xxxiv) hypothesis.] ∎

Lemma The assertion is true if A is the homeomorphic image of a closed ball.

For in this case, the inclusion $M - A \to M - x$, $x \in A$ is a homotopy equivalence: which follows from the special case $B - B_\epsilon \to B - 0$, where B is the closed unit ball in $\mathbb{R}^n$, and B_ϵ is the closed sub-ball of radius $\epsilon < 1$. It then follows from comparison

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(M) \arrow[r] \arrow[d, "="] & H_n(M, M-A) \arrow[r] \arrow[d, dashed, "\cong"] & H_{n-1}(M-A) \arrow[r] \arrow[d, "\cong"] & {} \\
\arrow[r] & H_n(M) \arrow[r] & H_n(M, M-x) \arrow[r] & H_{n-1}(M-x) &
\end{tikzcd}$$

of the exact sequences of the pairs $(M, M-A)$ and $(M, M-x)$ that j_A defines an isomorphism with the module of continuous functions from A to $\mathbb{Z}$. ∎

[§2.3 p viii*]

We can conclude the proof by recalling [that a smooth manifold has a <u>good cover</u>

(by geodesically convex sets, which are homeomorphic to Euclidean balls, and whose intersections are also geodesically convex). The assertion then follows by an application of Zorn's Lemma [Hatcher §3.3 p 248]. ∎

Remark This recourse to Riemannian geometry is quick, elegant, and unsatisfying, because the theorem is true without smoothness hypotheses, and can be proved by a slightly more complicated induction. I'll present that more precise argument in the proof of Poincaré duality below.

(8.3) To prove Poincaré duality for compact manifolds using the strategy above, we unfortunately need a version of the theorem true

for *non* compact manifolds; in particular, we need a generalization of

$$H^i_c(U) \xrightarrow{\cong} H_{n-i}(U)$$

to noncompact U, eg

$$H^i_c(\mathbb{R}^n) := H^i(\mathbb{R}^n_+, +) \cong H_{n-i}(\mathbb{R}^n) \cong \mathbb{Z} \quad \text{if } i=n$$

$$= 0 \quad \text{otherwise}$$

of the map defined by the cap-product with a fundamental class for U — in spite of the fact that $H_n(U) = 0$! Once we have such a map, we can compare the *cohomological* Mayer-Vietoris sequence

$$\to \tilde{H}^*(U_+ \cap V_+) \to \tilde{H}^*(U_+) \oplus \tilde{H}^*(V_+) \to H^*(U_+ \cup V$$

(ie

$$\to H^*(X, A \cup B) \to H^*(X,A) \oplus H^*(X,B) \to H^*(X, A \cap B.$$

defined by a pair (X,A), (X,B) with $A = X - U$, $B = X - V$,

xi)

via isomorphisms of the form

$$H_c^\bullet(U) = \tilde{H}^\bullet(U_+,+) \cong H^\bullet(X, X-U).$$

[Note that if $U \to X$ is an open embedding — a 1-1 map with image homeomorphic to U — then we have covariant homomorphisms

$$H_c^\bullet(U) \cong \tilde{H}^\bullet(X/(X-U)) \to H_c^\bullet(X)$$

(defined by $(X_+,+) \to (X/(X-U),+)$).]

with the homological Mayer-Vietoris sequence of U, V, and do an induction as above.

To define the missing cap product homomorphism we need another

__Lemma__ $\exists$ iso

$$H_c^\bullet(X) \longrightarrow \varinjlim_{K \text{compact} \subset X} H^\bullet(X, X-K).$$

xii)

__Proof__: Since direct limits preserve exactness, this follows from the exact sequence

$$0 \to \varinjlim_{K \text{cpct} \subset} S^\bullet(X, X-K) \to S^\bullet(X_+) \to \lim_{K \subset} S^\bullet(X-K) \to 0$$

But $\lim_{K \subset} S^\bullet(X-K) \cong S^\bullet(+)$ is the intersection of cochains on X_+ which vanish on chains whose support does not intersect $X-K$. ∎

To prove the general duality theorem we choose an orientation of M, ie a (generating) global section of the orientation sheaf; this restricts to define an element $[M]_K$ of $H_n(M, M-K)$ for any compact K, which extends to $[M]_{K'} \in H_n(M, M-K$ (since sections of the orientation sheaf are locally constant) if $K' \supset K$.

The relative cap product [§7.3 p xviii]

$$\begin{tikzcd}
H^i(M,M-K)\otimes \overset{[M]_K\in}{H_n(M,M-K)} \arrow[dd] \arrow[dr] & \\
 & H_{n-i}(M) \\
H^i(M,M-K')\otimes \overset{[M]_{K'}\in}{H_n(M,M-K')} \arrow[ur] &
\end{tikzcd}$$

is compatible with the restriction map on cohomology so taking direct limits defines the required map

$$\varinjlim [M]_K := D_M : H^i_c(M) \longrightarrow H_{n-i}(M),$$

whether M is compact or not.

As remarked above, it follows from Mayer-Vietoris sequence arguments that if U, V, and $U \cap V \subset M$ satisfy Poincaré duality, then so does $U \cup V$. Moreover it is not hard to see that if $\{U_i\}$ is a family of subsets of M, each satisfying Poincaré duality, with the family totally

ordered by inclusion, then their union satisfies duality.

Now it is clear from previous arguments that duality holds for coordinate neighborhoods which are images of convex balls in $\mathbb{R}^n$.

Suppose then that U is *any* coordinate patch, regarded as a subset of $\mathbb{R}^n$. *Enumerate* a dense set of points in U and choose a convex open $V_i \subset U$ containing the i^{th} point.

Let $U_1 = V_1$, $U_i = U_{i-1} \cup V_i$, $i > 1$.

The resulting family is closed under finite intersections, and the argument above implies that duality holds for U.

By Zorn's lemma there is a maximal open

subspace of M for which Poincaré duality holds. For any open V contained in a coordinate neighborhood, the theorem is true for $U \cup V$; so by maximality, $U = M$. ∎

§8.4 Applications

8.4.1 Intersection theory

Definition: The intersection product

$$\begin{tikzcd}
\bullet : H_i(M) \otimes H_k(M) \arrow[r, "\vec{D}_M \otimes \vec{D}_M"] & H^{n-i}(M) \otimes H^{n-k}(M) \arrow[dl, "\cup_M"] & \\
H^{2n-i-k}(M) \arrow[r, "D_M"] & H_{i+k-n}(M) &
\end{tikzcd}$$

makes the homology of a compact oriented manifold into a graded commutative ring.

For example, the intersection product of the class $[\mathbb{CP}_k]$ generating $H_{2k}(\mathbb{CP}^n)$ $(k \leq n)$ with the class $[\mathbb{CP}_j]$ generating $H_{2j}(\mathbb{CP}^n)$ is the generator $[\mathbb{CP}_{j+k-n}]$ of $H_{2(j+k-n)}(\mathbb{CP}^n)$, capturing the fact that the intersection of a generic $(k+1)$-dimensional subspace of $\mathbb{C}^{n+1}$ (defined by $n-k$ linear equations) with a $(j+1)$-dimensional subspace (defined by $n-j$ linear equations) is a $(j+k-n+1)$-dimensional linear subspace of $\mathbb{C}^{n+1}$.

[This intersection product is the historical source of the cap product symbol in this context, now used for a different (but closely related) construction. It inspired the use of the related cup product symbol for the product in cohomology

Example A map $f : X \to Y$ has an associated graph

$$\gamma(f)(x) = (x, f(x)) : X \longrightarrow X \times Y.$$

If $X = M$, $Y = N$ are compact oriented (m, resp. n-dimensional) manifolds, this defines

$$[\gamma(f)] \in H_m(M \times N)$$
$$= \gamma(f)_* [M]$$

(§3.8 p xxxix). The coincidence class $C(f,g)$ of two such maps $f, g : M \to N$ is the intersection product

$$\gamma(f) \circ \gamma(g) \in H_{m-n}(M \times N)$$

of their graph classes; for example if $\dim M = \dim N$ we get a class in $H_0(M \times N)$ which, if M and N are connected, can be regarded as an integral invariant of

of set $\gamma(f) \cap \gamma(g) = \{(x,y) \in M \times N \mid f(x) = \underset{=y}{g(x)}\}$

If f and g are smooth, these intersection (2.4 p xii) of their graphs will generically be a manifold of dimension $n-m$, with

$$\gamma(f) \cdot \gamma(g) = [\gamma(f) \cap \gamma(g)].$$

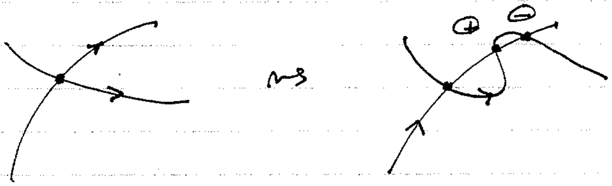
If for example $M = N$ and g is the identity map, then

$$\gamma(f) \cap \gamma(g) = \{(x,x) \in M \times M \mid x = f(x)\}$$

is (isomorphic to) the set of fixed points of f, and the Lefschetz fixed point theorem [§3.3 p xxii] identifies $[\gamma(f)] \cdot [\gamma(g)]$ geometrically (ie when $\gamma(f) \cap \gamma(g)$ is zero-dimensional) as the sum of (§3.8 p xxxviii) the fixed point indices of its components, and also algebraically, as the Lefschetz trace of the graded vector space homomorphism

$$f_* : H_*(X) \longrightarrow H_*(X).$$

This last step involves regarding f_* as an element of $H^*(X) \otimes H_*(X)$ [§7.3 p xviii] (with field coefficients), and thus (by Poincaré) as an element of degree n in $H_*(X) \otimes H_*(X)$: ie, as the class of $\gamma(f)$ in $H_*(X \times X)$.



8.4.2 The Thom isomorphism

From a more contemporary point of view, orientability of M is better understood as a property of its tangent bundle, eg equivalent to the triviality of its top

exterior power $\Lambda^n T(M) \cong M \times \mathbb{R}$; that a (connected) orientable manifold has two distinct orientations says that its frame bundle has two connected components, one being a principal (free) $Gl_n^+(\mathbb{R})$-bundle over M. Such characterizations make sense for general (not just tangent) bundles, and a more modern theory of orientability centers around the

Thom isomorphism theorem:

A vector bundle $V \to X$ (with X compact for simplicity) and fiber dimension n is orientable (with respect to ordinary coh $H^*(-,\mathbb{Z})$) if ∃ a Thom class

$$Th(V) \in H^n_c(V)$$

such that

xxi)

$$x \mapsto x \smile Th(V) : H^*(X) \xrightarrow{\cong} H_c^{*+n}(V)$$

is an isomorphism,

[Here the product is defined by identifying $H^*(X)$ with $H^*(B(X))$, where $B(V) \to X$ is the (homotopy-equivalence) of the closed ball bundle of V (eg defined in terms of a Riemannian metric), and by identifying $H_c^*(V^\ddagger)$ with $H^*(B(V), S(V))$, where $S(V) = \coprod_{x \in X} \partial B_x(V)$ is the bundle of spheres bounding the balls of $B(V)$. Thus $H_c^*(V) \cong H^*(B(V), S(V))$ is an $H^*(B(V)) \cong H^*(X)$-module]

Recall now that a smooth embedding

$$i : M \longrightarrow N$$

of manifolds ($n = \dim N > m = \dim M$) factors

xxii)

through a tubular neighborhood

$$M \to V(M) \subset N,$$

where $V(M) \to M$ is an open neighborhood of M in N, diffeomorphic (via the exponential map) to the ($(n-m)$-fiber dimensional) normal bundle $\nu(M)$ of M in N, ie

$$i^* T_N \cong \nu \oplus T_M, \text{ cf } [\S 2.4 \, p \, xi)].$$

The Pontryagin-Thom collapse map is the composition

$$N \to N / N - V(M) \cong \nu(M)^+ \quad ;$$

if $\nu(M)$ is orientable, this implies the existence of a commutative diagram

n−m = "codim" of

$$\begin{tikzcd} H^*(M) \arrow[r] \arrow[d, "D_M"] \arrow[rr, bend left, "i_!"] & H_c^{*+n-m}(\nu(M)) \arrow[r] & H^{*+n-m}(N) \arrow[d] \\ H_{m-*}(M) \arrow[rr, "i_*"] & & H_{m-*}(N) \end{tikzcd}$$

xxiii)

if M and N happen to be compact and orientable.

This construction can be vastly extended, because a map $M \to N$ between manifolds is homotopy equivalent to a composition

$$M \to N = N \times 0 \subset N \times \mathbb{R}^k, \quad k \gg 0,$$

which is homotopic, under very general conditions, to an embedding.

This implies the existence of a *bivariant* structure on the (co)homology of manifolds, ie the existence (under appropriate conditions) of *pairs* of homomorphisms

$$f^* : H^*(N) \to H^*(M) \quad \text{(an algebra homomorphism)}$$

$$f_! : H^*(M) \to H^{*+\mathrm{codim}}(N) \quad \text{(\underline{not} an algebra homomorphism)}$$

xxiv)

satisfying identities such as

$$(f \circ g)_! = f_! \circ g_!,$$

$$f_!(f^* x \cup y) = x \cup f_! y,$$

$$x \in H^* M, \quad y \in H^* N.$$

Pursuing these ideas leads to Grothendieck's version of the Riemann Roch theorem, the Atiyah-Singer index theorem, and other major intellectual developments.