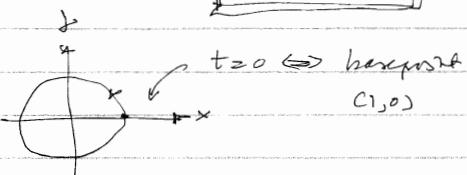


1)

§1 Smooth knots

615 Notes
1 Sept 2011

ii)

Defthe circle $S^1 =$  $t \geq 0 \Leftrightarrow$ horizontal

(1,0)

$$\mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^2$$

$$t \mapsto (\cos 2\pi t, \sin 2\pi t)$$

$$\left[s+t \Rightarrow k(s) \neq k(t) \right]$$

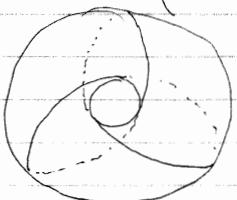
DefA continuous 1-1 fn. $k: S^1 \rightarrow \mathbb{R}^3$ is a (possibly wild knot).

Note, it's useful to think of k as a map to \mathbb{R}^3_+
 = one-point compactification of $\boxed{\mathbb{R}^3 \cong S^3}$

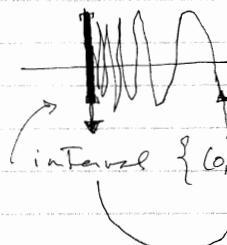
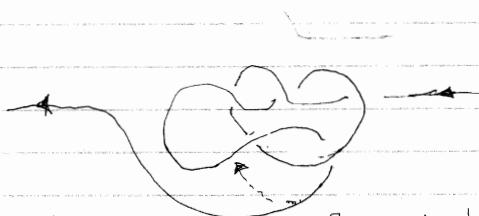
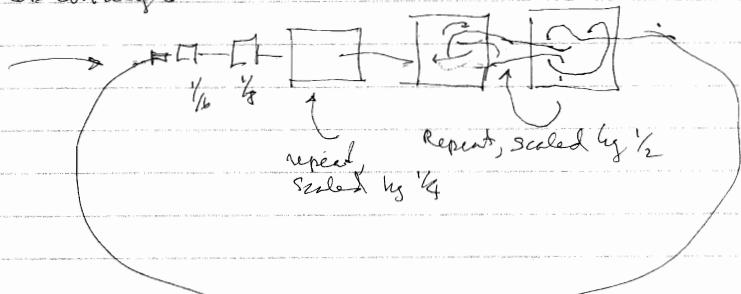
because then k maps a compact space
 to another compact space.

SKA
an embedding

Claim \exists subspace $\text{Emb}(S^1, \mathbb{R}^3_+)$ of
embeddings \subset space of continuous fns
 from S^1 to \mathbb{R}^3_+ , with the compact-open
 topology (eg eg Wikipedia for a defn.)
 or Hatcher p529 (Appendix)

the trefoil knot,drawn on a torus $\subset \mathbb{R}^3$.Ex

Ex A (2,5) torus knot.
 (why?)

graphed $\sin \frac{x}{2}$ (interval $\{(0,y) | y \in [0,1]\}$)Ex
Thisis NOT knotted: pull on it and it will
 disentangle.bad limit
point

⇒

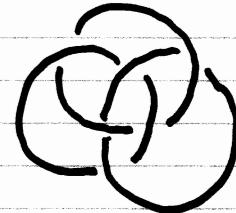
converges to a non
 differentiable fn,
 nevertheless an embedding.

i)

From now on I'll restrict to smooth one-to-one embeddings of S^1 in $\mathbb{R}^3_+ \cong S^3$

$\hookrightarrow \text{Emb}_{\text{smooth}}(S^1, \mathbb{R}^3_+) = \text{space of smooth knots in } \mathbb{R}^3$

Note \exists similar space of links



2 π_0 and π_1

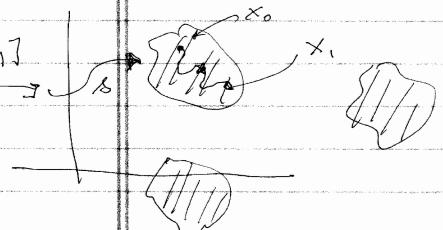
X not too nasty \in (Topological spaces)

\exists equivalence relation $x_0, x_1 \in X$

$$x_0 \sim x_1 \Leftrightarrow$$

\exists continuous function $s: [0, 1] \rightarrow X$,

$$s(0) = x_0, \quad s(1) = x_1,$$



Claim: $x_0 \sim x_0$ [identity]
 (use the constant fn.
 $s(t) = x_0, t \in [0, 1]$)

$x_0 \sim x_1 \Rightarrow x_1 \sim x_0$ [symmetry]
 (path $s \mapsto$ path $\tilde{s}(t) = s(1-t)$,
 run s backwards.)

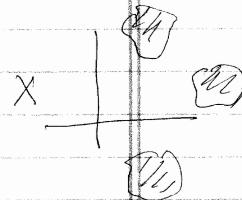
$x_0 \sim x_1, \quad x_1 \sim x_2$
 $\Rightarrow x_0 \sim x_2$

[transitivity]

ii)

Defn $\pi_0(X) = \text{set of equivalence classes}$
 of points of X

= set of path components of X

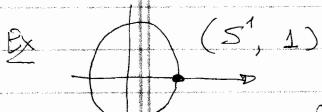


$\{\cdot\} \quad \pi_0(X) = \text{set with 3 elements!}$

Defn Suppose X is a (not too nasty) path-connected space: $\pi_0(X) = \{\cdot\}$ has one element.

Defn There is a path between any two $x_0, x_1 \in X$.

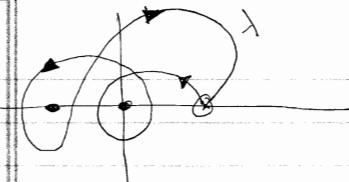
A pointed space (X, x) is a pair consisting of
 a space X and a basepoint $x \in X$.



A map $f: (X, x) \rightarrow (Y, y)$ of
 pointed spaces is a continuous function
 $f: X \rightarrow Y$ such that $f(x) = y$.
 (base)

Defn A loop in a (path-connected) pointed space (X, x)
 is a map $\lambda: (S^1, 1) \rightarrow (X, x)$

v)



A loop in

$$\mathbb{R}^2 - \{(0,0), (-1,0)\}$$

based at $(1,0)$.

Defn

$$\Omega(X, x) = \text{Maps}(S^1, 1), (X, x)$$

= loop space of X = space of loops in X ,
based at $x \in X$.

Defn

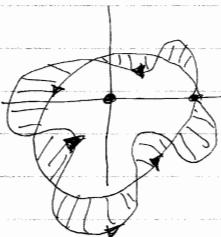
 $\pi_1(X, x)$ = the fundamental group of
 X , based at x

$$= \pi_0 \Omega(X, x)$$

(base)

= set of equivalence classes of loops in X

Ex

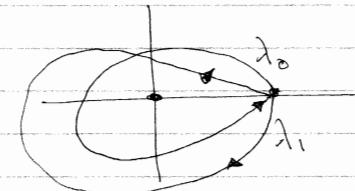


$$\mathbb{R}^2 - \{(0,0)\} = X$$

these two loops represent
the same elts. of
 $\pi_1(\mathbb{R}^2 - \{\text{one pt}\})$ Note,
the defn of π_1 from π_0 can be
extended to define π_2, π_3, \dots .

vi)

Important FACTS

1) $\pi_1(X, x)$ is a group:each loop has an inverse; there is an identity loop (the constant map $S^1 \rightarrow (X, x)$ which sends every $s \in S^1$ to x); there is a composition operator on loops $\lambda_1 \cdot \lambda_0 = \text{first do } \lambda_0, \text{ then do } \lambda_1$.Claim this defines an associative composition
law on π_1 ...

Reference

Hatcher §1.1 Prop 1.3 p26,

or last chapter of Munkres' Topology

Ex

$$\pi_1(S^1, 1) \cong \pi_1(\mathbb{R}^2 - \{\text{pt}\}, \{\text{pt}\}) \cong \mathbb{Z}$$

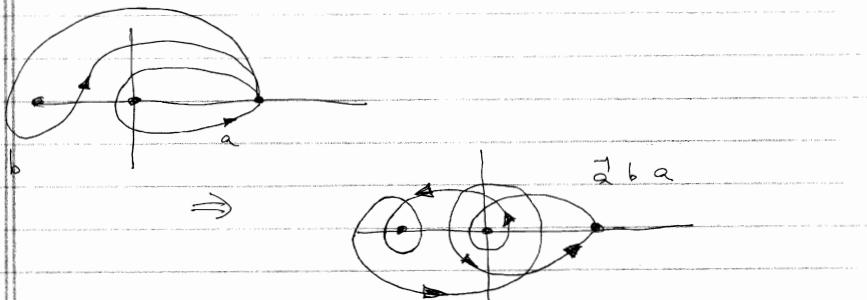
$$\cong \pi_1(\text{plane annulus})$$

Hatcher,
ch 0 p 3because these spaces are all homotopy-equivalent!

vii)

Ex.

$$\pi_1(\mathbb{R}^2 - \{\text{2 points}\}) = \text{free group on 2 generators}$$



Any "worst" $a^{x_1} b^{y_1} a^{x_2} b^{y_2} \dots a^{x_n} b^{y_n}$
with $x_1, x_2, \dots, y_1, y_2, \dots, y_n \in \mathbb{Z}$
represents the equivalence class of some loop γ ,
and two such loops are equivalent iff
they have the same defining sequence of
integers.

Defn $F_n = \text{free group on } n \text{ generators}$
= $\langle x_1, \dots, x_n \mid \dots \rangle$ and no relations
 $N(r)$

Ex If collection $\{r_1, \dots, r_k\} \subseteq F_n$ \exists smallest normal
subgroup of F_n containing $\{r_1, \dots, r_k\}$.

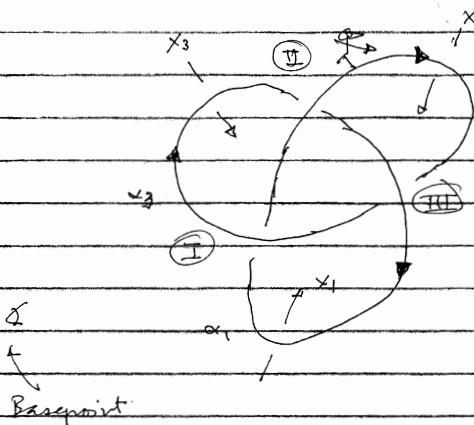
$$F_n / N(r) = \text{quotient group} := \langle x_1, \dots, x_n \mid r_1 = \dots = r_k = 1 \rangle$$

group generated by x_1, \dots, x_n , with relations r_1, \dots, r_k .

$$\text{Ex } F_2 / N(ab\bar{a}\bar{b}) = \mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle \text{ 2 generators}$$

iii)

6) orient the mesh

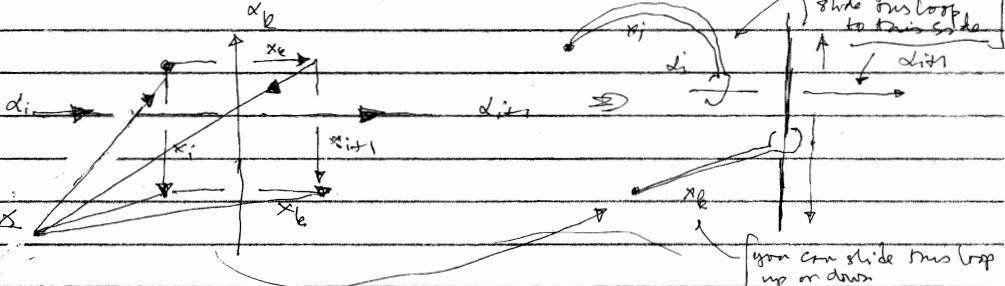


i) Fix a projection
with at worst
double points

ii) Label the resulting
and consecutive

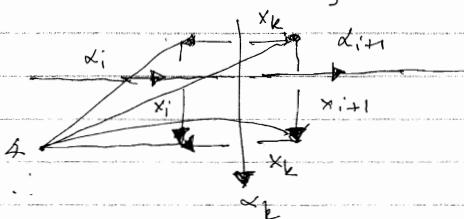
iii) assign undercrossings,
oriented by the
right-hand rule

claim Each undercrossing x_k defines the homotopy
class of a loop in $\pi_1(\mathbb{R}^3 - k, *)$: but you can't



These loops satisfy relations associated to the crossings:

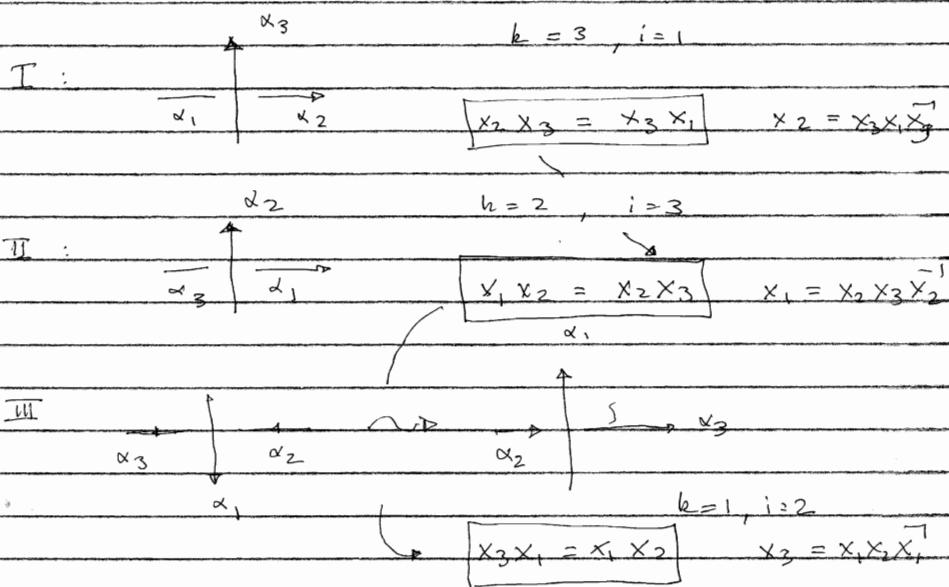
$$x_{i+1} x_k = x_k x_i, \quad x_i = x_k^{-1} x_{i+1} x_k$$



$$x_i x_k = x_k x_{i+1},$$

$$x_i = x_k x_{i+1} x_k^{-1}$$

For Ex. the braid has three crossings:



where $i <$ consequence of the preceding relations

[This is a general fact: the Wintgen presentation for a knot diagram with n crossings has n generators and n relations, but any one of these relations is a consequence of the others.]

If we write $x_1 = x, x_2 = y, x_3 = z$ we get

$$y = z \bar{z}, \quad x = y \bar{z} \bar{y}, \quad z = x \bar{y} \bar{x}.$$

x)

This presentation can be simplified:

$$\text{Set } z = xy\bar{x} \text{ in } y = z \bar{z}$$

$$\therefore y = : (xy\bar{x}) \cdot x \cdot (xy\bar{x})^{-1}$$

$$= \underbrace{xy\bar{x}}_{\text{cancel}} \cdot x \cdot \bar{x} = xy\bar{x}x = xyx\bar{x}\bar{x}$$

Multiply both sides on right by xy :

$$\therefore yxy = xy\bar{x}\bar{x}\bar{x}y = xyx$$

$$\Rightarrow \pi_1(\mathbb{R}^3 - \text{braid}) = \langle x, y \mid xyx = yxy \rangle$$

[Alternatively: set $z = xy\bar{x}$ in $x = y \bar{z} \bar{y}$:

$$\therefore x = y \cdot xy\bar{x} \cdot \bar{y}; \text{ multiplying by } yx \text{ on right:}$$

$$xyx = y \cdot xy\bar{x}\bar{y} \cdot yx$$

$$\Rightarrow xgx = yxy, \text{ as we saw before.}$$

A)

Homework:

1) Let $G = \langle a, b \mid a^2 = b^3 \rangle$

Show that the maps

$$\pi_1(\mathbb{R}^3 - \text{trefoil}) \xrightarrow{\phi} G$$

defined by $\phi(x) = b\bar{a}^{-1}b$, $\phi(y) = \bar{b}^2a$
 $\phi(a) = xy^2$, $\phi(b) = xy$

are inverse homomorphisms, so their groups are
isomorphic.

2) Calculate the fundamental group associated
to the diagram

