Problem Set I for 110.615:

In class [week of 1/9/011] we calculated the fundamental group

$$\pi_1(\mathbb{R}^3_+ - K, +) \;\cong\; \langle x, y \mid xyx = yxy \rangle$$

of the complement of the trefoil knot K, and in the following week we calculated the fundamental group

$$\pi_1(\mathrm{Config}^n(\mathbb{C})/\Sigma_n, *) \;\cong\; \langle \sigma_1, \cdots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle$$

of the space of n distinct **unordered** points in the plane. [The basepoint $*$ is the unordered configuration $\{1, \ldots, n\} \subset \mathbb{R} \subset \mathbb{C}$.]

When $n = 3$ these groups are isomorphic (via $\sigma_1 \mapsto x,\ \sigma_2 \mapsto y$). In fact there is a nice map (due to Quillen, but written up by Milnor)

$$\mathbf{z} \mapsto \mathbf{q} : \mathrm{Config}^n(\mathbb{C})/\Sigma_n \to \mathbb{R}^3_+ - K \;\subset\; S^1 \times S^1 \subset S^3$$

from the space of three distinct unordered points in the plane, to the complement of the trefoil, drawn as a knot on a torus in the three-sphere.

This problem set sketches the construction of this map [which is in fact a homotopy equivalence. It can't be a homeomorphism, since its domain is a six-dimensional real manifold, while its range is three-dimensional. In fact $\mathbf{q}$ is the projection of a fiber bundle, the quotient by an action of the affine group

$$\mathrm{Aff} = \{(\lambda, w) \in \mathbb{R}^\times_+ \rtimes \mathbb{C} \mid z \mapsto \lambda z + w\}$$

on the space of configurations ...]

At places where the argument seems straightforward, I've replaced some details with the symbol $\bullet$?; your mission (should you choose to accept it) is to fill in these dots. Please get your answers to me by e-mail [a single .pdf file, please; scans of handwritten writeups are perfectly acceptable] by 6 AM Saturday 1 October.

§II Configuration spaces and symmetric functions

Definition:

$$\mathbf{z} = (z_1, \ldots, z_n) \in \mathrm{Config}^n(\mathbb{C}) \text{ iff } i \neq k \Rightarrow z_i \neq z_k \ ,$$

is the space of n distinct **labeled** points in the plane $\mathbb{R}^2 \cong \mathbb{C}$; alternately,

$$\mathrm{Config}^n(\mathbb{C}) \ = \ \mathbb{C}^n - \bigcup_{i,k} \{\mathbf{z} \in \mathbb{C}^n \mid z_i = z_k\} \ .$$

The map

$$\mathbf{z} \mapsto \mathbf{e} : \mathrm{Config}^n(\mathbb{C}) \to \mathbb{C}^n$$

defined by

$$\prod_{1 \leq i \leq n} (t - z_i) \ := \ p_{\mathbf{z}}(t) = t^n - e_1 t^{n-1} + e_2 t^{n-2} - \cdots \pm e_n \in \mathbb{C}[t] \ ,$$

goes back to Newton; it sends the configuration $\mathbf{z}$ to the monic polynomial with the z_i's as its roots. The e_k's are the elementary symmetric function of the **unordered** set of coordinates z_i:

$$e_1 = \bullet\ ?, \ e_2 = \bullet\ ?, \ e_n = \bullet\ ? \ ,$$

so this construction factors through a map

$$\mathrm{Config}^n(\mathbb{C})/\Sigma_n \to \mathbb{C}^n$$

defined on the space of n distinct **un**labelled points in the plane.

Observe that if $t = T + \frac{1}{n} e_1$ then

$$p_{\mathbf{z}}(t) = \sum_{0 \leq k \leq n} (-1)^{n-k} e_{n-k} t^k \mapsto T^n + \sum_{0 \leq k \leq n-2} (-1)^{n-k} g_{n-k} T^k$$

defines a map

$$\mathbf{e} \mapsto \mathbf{g} : \mathbb{C}^n \to \mathbb{C}^{n-1} \ ;$$

for example, if $n = 3$ then

$$(e_1, e_2, e_3) \mapsto \bullet\ ? \ .$$

Note, if $\lambda \in \mathbb{C}^\times$ and $\mathbf{z} \mapsto \lambda \mathbf{z}$ then

$$e_k \mapsto \bullet\ ?, \ g_k \mapsto \bullet\ ? \ .$$

§III The discriminant locus

Definition, the **discriminant** of $\mathbf{z}$ is

$$\Delta_(\mathbf{z}) = (-1)^n \prod_{i \neq k} (z_i - z_k) = \prod_{i > k} (z_i - z_k)^2 \in \mathbb{C}[\mathbf{e}] \; .$$

The discriminant is symmetric, and hence is expressible in terms of the $\mathbf{e}$'s:

$$\Delta(z_1, z_2) = \bullet \; ? \; \Delta(z_1, z_2, z_3) = \bullet \; ? \; .$$

Proposition: There is a homeomorphism

$$\mathrm{Config}^n(\mathbb{C})/\Sigma_n \to \mathbb{C}^n - \Delta^{-1}(0)$$

where the **discriminant locus**

$$\Delta^{-1}(0) = \{\mathbf{z} \in \mathbb{C}^n \mid \Delta(\mathbf{z}) = 0\}$$

can be identified with the set of polynomials $p_{\mathbf{z}}$ with **repeated roots**.

Note that

$$p_{\mathbf{z}}(t) = \prod_{1 \leq i \leq n} (t - z_i) = \prod_{1 \leq i \leq n} (T - \tilde{z}_i) \; ,$$

where

$$\tilde{z}_i = z_i - \frac{1}{n}(z_1 + \cdots + z_n) \; ;$$

Note also that $\sum t z_i = 0$, and that if $z_i \neq z_k$ then $\tilde{z}_i \neq \tilde{z}_k$, so the $\tilde{z}$'s are what physicists call 'center-of-mass' coordinates. Note finally that the affine translation $\mathbf{z} \mapsto \mathbf{z} + w \cdot (1, \ldots, 1)$ doesn't change the discriminant.

If $n \geq 3$ define the **reduced** discriminant to be $D(\mathbf{g}) = \Delta(\mathbf{e})$, eg

$$D_3 = \bullet \; ? \; .$$

This defines a map

$$\mathbf{e} \mapsto \mathbf{g} : \mathbb{C}^n - \Delta^{-1}(0) \to \mathbb{C}^{n-1} - D^{-1}(0) \; .$$

§IV Quillen's normalization

Proposition: If $a, b \geq 0$ are not both 0, then there is a unique $x(a, b) > 0 \in \mathbb{R}$ such that

$$ax^2 + bx^3 = 400 .$$

Proof: The function $h(x) = ax^2 + bx^3 - 400$ has derivative $h'(x) = 2ax + 3bx^2 > 0$ if $x > 0$, so $h(x)$ is increasing on the positive real line. Since h is negative at $x = 0$, and positive when x is large, its graph crosses the x-axis at a unique point $x(a, b)$.

Moreover, the implicit function theorem tells us that $x = x(a, b)$ satisfies the equations

$$\frac{\partial x}{\partial a} = \bullet\ ?, \ \frac{\partial x}{\partial b} = \bullet\ ?$$

and since $2a + 3bx = \bullet\ ? > 0$, $(a, b) \mapsto x(a, b)$ is a smooth function (on the first quadrant of the (a, b)-plane).

Define $\mathbf{g} \mapsto \mathbf{q} : \mathbb{C}^2 - D^{-1}(0) \to S^3(20)$ by

$$(g_2, g_3) \mapsto (xg_2, x^{3/2}g_3) ,$$

where $x := x(|g_2|^2, |g_3|^2)$ is as in the proposition above. Since $x > 0$ [$\bullet$?] and

$$|\mathbf{q}(g_2, g_3)|^2 = \bullet\ ? = 20^2 ,$$

this function takes values in the 3-dimensional sphere of radius 20 in $\mathbb{C}^2$.

[**Check** $\bullet$? that if $Z_i = \lambda z_i + w$, with $(\lambda, w) \in \mathrm{Aff}$, then $\mathbf{q}(\mathbf{g}(\mathbf{Z})) = \mathbf{q}(\mathbf{g}(\mathbf{z}))$.]

Moreover, we have

$$4q_1^3 + 27q_2^2 = \bullet\ ? \cdot D \neq 0 ,$$

so $\mathbf{q}$ maps the complement of the reduced discriminant locus to the **complement** in $S^3(20)$ of the set

$$K = \{(q_1, q_2) \in \mathbb{C}^2 \mid |q_1|^2 + |q_2|^2 = 20^2,\ 4q_1^3 + 27q_2^2 = 0\} .$$

This is the zero-set of three equations in four-dimensional space, which we expect to be one-dimensional. In fact K is parametrized by

$$t \mapsto (\bullet\ ? \exp(\bullet\ ?t), \bullet\ ? \exp(\bullet\ ?t)) \in \mathbb{C}^2 ,$$

which lies on the torus $S^1(12) \times S^1(16) \subset S^3(20)$ defined by the product of two circles, one of radius 12, the other of radius 16. The projection

$$S^1 = \mathbb{R}/2\pi\mathbb{Z} \ni t \mapsto q_1(t) \in S^1(12)$$

has winding number **three**, while

$$S^1 = \mathbb{R}/2\pi\mathbb{Z} \ni t \mapsto q_2(t) \in S^1(16)$$

has winding number **two**; thus K is a $(2,3)$-torus knot, ie the trefoil.