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Hölder continuity of the solutions for a class of nonlinear SPDE's arising from one dimensional superprocesses

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Abstract The Hölder continuity of the solution $X_t(x)$ to a nonlinear stochastic partial differential equation (see (1.2) below) arising from one dimensional superprocesses is obtained. It is proved that the Hölder exponent in time variable is arbitrarily close to $1/4$, improving the result of $1/10$ in Li et al. (to appear on Probab. Theory Relat. Fields.). The method is to use the Malliavin calculus. The Hölder continuity in spatial variable x of exponent $1/2$ is also obtained by using this new approach. This Hölder continuity result is sharp since the corresponding linear heat equation has the same Hölder continuity.

Keywords Nonlinear stochastic partial differential equation · Stochastic heat kernel · Conditional transition probability density in a random environment · Malliavin calculus · Hölder continuity · Moment estimates

Mathematics Subject Classification 60H07 · 60H15 · 60H30

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1 Introduction

Consider a system of particles indexed by multi-indexes α in a random environment whose motions are described by

$$x_\alpha(t) = x_\alpha + B^\alpha(t) + \int_0^t \int_{\mathbb{R}} h\left(y - x_\alpha(u)\right) W\left(du, dy\right), \tag{1.1}$$

where $h \in L^2(\mathbb{R})$, $(B^\alpha(t); t \geq 0)_\alpha$ are independent Brownian motions and W is a Brownian sheet on $\mathbb{R}_+ \times \mathbb{R}$ independent of B^α. For more detail about this model, we refer to Wang ([10,11]) and Dawson et al. [2]. Under some specifications for the branching mechanism and in the limiting situation, Dawson et al. [3] obtained that the density of the branching particles satisfies the following stochastic partial differential equation (SPDE):

$$X_t(x) = \mu(x) + \int_0^t \Delta X_u(x) du - \int_0^t \int_{\mathbb{R}} \nabla_x \left(h\left(y - x\right) X_u\left(x\right)\right) W\left(du, dy\right)$$
$$+ \int_0^t \sqrt{X_u\left(x\right)} \frac{V\left(du, dx\right)}{dx}, \tag{1.2}$$

where V is a Brownian sheet on $\mathbb{R}_+ \times \mathbb{R}$ independent of W. The joint Hölder continuity of $(t, x) \longmapsto X_t(x)$ was left as an open problem in [3].

Let $H_2^k(\mathbb{R}) = \left\{u \in L^2(\mathbb{R});\ u^{(i)} \in L^2(\mathbb{R}) \, for \, i = 1, 2, \ldots, k\right\}$, the Sobolev space with norm $\|h\|_{k,2}^2 = \sum_{i=0}^k \left\|h^{(k)}\right\|_{L^2(\mathbb{R})}^2$. In a recent paper, Li et al. [4] proved that $X_t(x)$ is almost surely jointly Hölder continuous, under the condition that $h \in H_2^2(\mathbb{R})$ with $\|h\|_{1,2}^2 < 2$ and $X_0 = \mu \in H_2^1(\mathbb{R})$ is bounded. More precisely, they showed that for any $\epsilon > 0$, $X_t(x)$ is Hölder continuous in x with exponent $1/2 - \epsilon$ and in t with exponent $1/10 - \epsilon$. Comparing to the Hölder continuity for the stochastic heat equation which has the Hölder continuity of $1/4 - \epsilon$ in time, it is conjectured that the Hölder continuity of $X_t(x)$ should also be $1/4 - \epsilon$.

The aim of this paper is to provide an affirmative answer to the above conjecture. Here is the main result of this paper.

Theorem 1.1 *Suppose that $h \in H_2^2(\mathbb{R})$ and $X_0 = \mu \in L^2(\mathbb{R})$ is bounded. Then for any $p \geq 1$ and $T > 0$, there exists a constant C depending on p, T, $\|h\|_{2,2}$ and $\|\mu\|_{L^2(\mathbb{R})}$ such that for any $x, y \in \mathbb{R}$ and $0 \leq s < t \leq T$,*

$$E\left|X_t\left(y\right) - X_s\left(x\right)\right|^{2p} \leq Ct^{-p}(|x - y|^{p-\frac{1}{2}} + (t - s)^{\frac{p}{2}-\frac{1}{4}}). \tag{1.3}$$

Moreover, if μ is also Hölder continuous with exponent $\lambda > \frac{1}{2}$, then

$$E\left|X_t\left(y\right) - X_s\left(x\right)\right|^{2p} \leq C(1 + \|\mu\|_\lambda)(|x - y|^{p-\frac{1}{2}} + (t - s)^{\frac{p}{2}-\frac{1}{4}}). \tag{1.4}$$

We remark here that the term t^{-p} in the right hand side of (1.3) implies that the Hölder norm of $X_t(x)$ blows up as $t \to 0$. This problem arises naturally since we only assume $X_0 = \mu \in L^2(\mathbb{R})$. By assuming that μ is Hölder continuous as well, we eliminate this singularity in (1.4).

Applying an anisotropic version of the Kolmogorov's continuity criteria for parabolic metric (see [1], Corollary A.3; see also [7], p. 31) we can obtain the following Hölder continuity result:

Corollary 1.2 *Under the assumptions of Theorem 1.1, $X_t(x)$ has a Hölder continuous version (still denoted by $X_t(x)$). More precisely, if $X_0 = \mu \in L^2(\mathbb{R})$ is bounded, then for any $0 < \delta < T$, $K > 0$ and $\alpha \in (0, \frac{1}{2})$, there exists a random variable $G_{T,K,\alpha} \geq 0$ with $E[G_{T,K,\alpha}] < \infty$ such that a.s.*

$$|X_t(x) - X_s(y)| \leq \delta^{-\frac{1}{2}} G_{T,K,\alpha} \left(|t-s|^{\frac{1}{2}} + |x-y| \right)^{\alpha}, \tag{1.5}$$

for all $\delta \leq s < t \leq T$, $x, y \in [-K, K]$. Furthermore, if μ is also Hölder continuous with exponent $\lambda > \frac{1}{2}$, then

$$|X_t(x) - X_s(y)| \leq G_{T,K,\alpha} \left(|t-s|^{\frac{1}{2}} + |x-y| \right)^{\alpha}. \tag{1.6}$$

When $h = 0$ Eq. (1.2) is reduced to the famous Dawson–Watanabe equation (process). The joint Hölder continuity for this equation has been studied by Konno and Shiga [5] and Reimers [9]. The starting point is to interpret the equation (when $h = 0$) in mild form with the heat kernel associated with the Laplacian Δ in (1.2). Then the properties of the heat kernel (Gaussian density) can be fully used to analyze the Hölder continuity.

The straightforward extension of the mild solution concept and technique to general nonzero h case in (1.2) meets a substantial difficulty. To overcome this difficulty, Li et al. [4] replace the heat kernel by a random heat kernel associated with

$$\int_0^t \Delta X_u(x) dr - \int_0^t \int_{\mathbb{R}} \nabla_x (h(y-x) X_u(x)) W(du, dy).$$

The random heat kernel is given by the conditional transition function of a typical particle in the system with W given. To be more precise, consider the spatial motion of a typical particle in the system:

$$\xi_t = \xi_0 + B_t + \int_0^t \int_{\mathbb{R}} h(y - \xi_u) W(du, dy), \tag{1.7}$$

where $(B_t; t \geq 0)$ is a Brownian motion. For $r \leq t$ and $x \in \mathbb{R}$, define the conditional (conditioned by W) transition probability by

$$P_t^{r,x,W}(\cdot) \equiv P^W(\xi_t \in \cdot | \xi_r = x). \tag{1.8}$$

Denote by $p^W(r,x;t,y)$ the density of $P_t^{r,x,W}(\cdot)$. It is proved that $X_t(y)$ has the following convolution representation:

$$X_t(y) = \int_{\mathbb{R}} \mu(z) p^W(0,z;t,y)\,dz + \int_0^t \int_{\mathbb{R}} p^W(r,z;t,y)\,Z(dr,dz) \equiv X_{t,1}(y) + X_{t,2}(y), \tag{1.9}$$

where $Z(dr,dz) = \sqrt{X_r(z)}V(dr,dz)$. Then they introduce a fractional integration by parts technique to obtain the Hölder continuity estimates, using Krylov's L_p theory (cf. Krylov [6]) for linear SPDE.

In this paper, we shall use the techniques from Malliavin calculus to obtain more precise estimates for the conditional transition function $p^W(r,x;t,y)$. This allows us to improve the Hölder continuity in the time variable for the solution $X_t(x)$.

The rest of the paper is organized as follows: In Sect. 2, we briefly recall some notations and results on Malliavin calculus. Then we derive moment estimates for the conditional transition function in Sect. 3. We study the Hölder continuity in spatial and time variables of $X_t(x)$ in Sects. 4 and 5 respectively. The proof of Theorem 1.1 is concluded in Sect. 5.

Along the paper, we shall use the following notation: $\|\cdot\|_H$ denotes the norm on Hilbert space $H = L^2([0,T])$, $\|\cdot\|$ (and $\|\cdot\|_p$) denotes the norm on $L^2(\mathbb{R})$ (and on $L^p(\Omega)$). The expectation on $(\Omega, \mathcal{F}, P)$ is denoted by E and the conditional expectation with respect to the process W is denoted by E^B.

We denote by C a generic positive constant depending only on p, T, $\|h\|_{2,2}$ and $\|\mu\|_{L^2(\mathbb{R})}$.

2 Preliminaries

Fix a time interval $[0,T]$. Let $(B_t; t \geq 0)$ be a standard Brownian motion. Let $\mathcal{S}$ denote the class of smooth random variables of the form $F = f(B_{t_1}, \ldots, B_{t_n})$, where $t_1, \ldots, t_n \in [0,T]$, $n \geq 1$, and $f \in C_p^\infty(\mathbb{R}^n)$, the set of smooth functions f such that f itself and all its partial derivatives have at most polynomial growth. Given $F = f(B_{t_1}, \ldots, B_{t_n})$ in $\mathcal{S}$, its Malliavin derivative DF is the H–valued ($H = L^2([0,T])$) random variable given by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B_{t_1}, \ldots, B_{t_n}))\mathbf{1}_{[0,t_i]}(t).$$

The derivative operator D is a closable and unbounded operator on $L^2(\Omega)$ taking values in $L^2(\Omega, H)$. For any $p \geq 1$, we denote by $\mathbb{D}^{1,p}$ the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{1,p}$ given by:

$$\|DF\|_{1,p}^p = E(|F|^p) + E(\|DF\|_H^p).$$

We denote by δ the adjoint operator of D, which is unbounded from a domain in $L^2(\Omega, H)$ to $L^2(\Omega)$. In particular, if $u \in Dom(\delta)$, then $\delta(u)$ is characterized by the following duality relation:

$$E(\delta(u)F) = E(\langle DF, u\rangle_H) \quad \textit{for any } F \in \mathbb{D}^{1,2}.$$

The operator δ is called the *divergence* operator. The following two lemmas are from [8], Propositions 1.5.4 and 2.1.1 and are used frequently in this paper.

Lemma 2.1 *The divergence operator δ is continuous from $\mathbb{D}^{1,p}(H)$ to $L^p(\Omega)$, for any $p > 1$. That is, there exists a constant C_p such that*

$$\|\delta(u)\|_{L^p(\Omega)} \le C_p \left(\|Eu\|_H + \|Du\|_{L^p(\Omega, H\otimes H)}\right). \tag{2.1}$$

Lemma 2.2 *Let F be a random variable in the space $\mathbb{D}^{1,2}$, and suppose that $\frac{DF}{\|DF\|_H^2}$ belongs to the domain of the operator δ in $L^2(\Omega)$. Then the law of F has a continuous and bounded density given by*

$$p(x) = E\left[\mathbf{1}_{\{F>x\}}\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right].$$

From Lemma 2.2 we can deduce estimates for the density.

Lemma 2.3 *Let F be a random variable and let $u \in \mathbb{D}^{1,q}(H)$ with $q > 1$. Then for the conjugate pair p and q (i.e. $\frac{1}{p} + \frac{1}{q} = 1$) and for all $x \in \mathbb{R}$,*

$$\left|E\left[\mathbf{1}_{\{F>x\}}\delta(u)\right]\right| \le (P(|F| \ge |x|))^{\frac{1}{p}} \|\delta(u)\|_{L^q(\Omega)}. \tag{2.2}$$

Proof If $x > 0$, we have $E\left[\mathbf{1}_{\{F>x\}}\right] = P(F > x) \le P(|F| \ge |x|)$. Then (2.2) follows from Hölder's inequality. If $x \le 0$, we have $E\left[\mathbf{1}_{\{F\le x\}}\right] = P(F \le x) \le P(|F| \ge |x|)$. From the definition of δ, we have $E\left[\left(\mathbf{1}_{\{F>x\}} + \mathbf{1}_{\{F\le x\}}\right)\delta(u)\right] = E\delta(u) = 0$. Then, $E\left[\mathbf{1}_{\{F>x\}}\delta(u)\right] = -E\left[\mathbf{1}_{\{F\le x\}}\delta(u)\right]$. Then (2.2) follows again from Hölder's inequality.

3 Moment estimates

In this section, we derive moment estimates for the derivatives of ξ_t and the conditional transition function $p^W(r, x; t, y)$.

Recall that $\xi_t = \xi_t^{r,x}$ with initial value $\xi_r = x$ is given by

$$\xi_t = x + B_r^t + I_r^t(h), \quad 0 \le r < t \le T, \tag{3.1}$$

where we introduced the notations

$$B_r^t \equiv B_t - B_r, \text{ and } \quad I_r^t(h) \equiv \int_r^t \int_{\mathbb{R}} h(y - \xi_u) W(du, dy). \tag{3.2}$$

Since $h \in H_2^2(\mathbb{R})$, by using the standard Picard iteration scheme, we can prove that such a solution ξ_t to the stochastic differential equation (3.1) exists, and by a regularization argument of h we can prove that $\xi_t \in \mathbb{D}^{2,2}$ (here the Malliavin derivative is with respect to B). Taking the Malliavin derivative D_θ with respect to B, we have

$$D_\theta \xi_t = \mathbf{1}_{[r,t]}(\theta) \left[1 - \int_\theta^t \int_{\mathbb{R}} h'(y - \xi_u) D_\theta \xi_u W(du, dy) \right]. \tag{3.3}$$

Note that

$$M_{\theta,t} := \int_\theta^t \int_{\mathbb{R}} h'(y - \xi_u) W(du, dy)$$

is a martingale with quadratic variation $\langle M \rangle_{\theta,t} = \|h'\|^2 (t - \theta)$ for $t > \theta$. Thus

$$D_\theta \xi_t = \mathbf{1}_{[r,t]}(\theta) \exp\left(M_{\theta,t} - \frac{1}{2} \|h'\|^2 (t - \theta) \right). \tag{3.4}$$

As a result, we have

$$D_\eta D_\theta \xi_t = \mathbf{1}_{[r,t]}(\theta) \exp\left(M_{\theta,t} - \frac{1}{2} \|h'\|^2 (t - \theta) \right) D_\eta M_{\theta,t} = D_\theta \xi_t \cdot D_\eta M_{\theta,t}, \tag{3.5}$$

where $D_\eta M_{\theta,t} = \mathbf{1}_{[\theta,t]}(\eta) \int_\eta^t \int_{\mathbb{R}} h''(y - \xi_u) D_\eta \xi_u W(du, dy)$.

The next lemma gives estimates for the moments of $D\xi_t$ and $D^2\xi_t$.

Lemma 3.1 *For any* $0 \le r < t \le T$ *and* $p \ge 1$*, we have*

$$\| \|D\xi_t\|_H \|_{2p} \le \exp\left((2p - 1) \|h'\|^2 (t - r) \right) (t - r)^{\frac{1}{2}}, \tag{3.6}$$

$$\left\| \left\| D^2 \xi_t \right\|_{H \otimes H} \right\|_{2p} \le C_p \|h''\| \exp\left((4p - 1) \|h'\|^2 (t - r) \right) (t - r)^{\frac{3}{2}}, \tag{3.7}$$

and for any $\gamma > 0$,

$$E(\|D\xi_t\|_H^{-2\gamma}) \le \exp\left(\left(2\gamma^2 + \gamma\right) \|h'\|^2 (t - r) \right) (t - r)^{-\gamma}. \tag{3.8}$$

Proof Note that for any $p \geq 1$ and $r \leq \theta < t$,

$$\|D_\theta \xi_t\|_{2p}^2 = \left(E \exp\left[2p\left(M_{\theta,t} - \frac{1}{2}\|h'\|^2 (t-\theta)\right)\right]\right)^{\frac{1}{p}}$$
$$= \exp\left((2p-1)\|h'\|^2 (t-\theta)\right). \tag{3.9}$$

Then (3.6) follows from Minkowski's inequality and (3.9) since

$$\left\| \|D\xi_t\|_H \right\|_{2p}^2 = \left[E\left(\int_r^t |D_\theta \xi_t|^2 d\theta\right)^p\right]^{\frac{1}{p}} \leq \int_r^t \|D_\theta \xi_t\|_{2p}^2 d\theta.$$

Applying the Burkholder-Davis-Gundy inequality we have for $r \leq \theta \leq \eta < t$

$$\left\|D_\eta M_{\theta,t}\right\|_{2p}^2 \leq C_p \left(E\left|\int_\eta^t \int_{\mathbb{R}} \left|h''(y-\xi_u) D_\eta \xi_u\right|^2 dy du\right|^p\right)^{\frac{1}{p}}$$
$$\leq C_p \left\|h''\right\|^2 \int_\eta^t \left\|D_\eta \xi_u\right\|_{2p}^2 du. \tag{3.10}$$

Combining (3.5), (3.9) and (3.10) yields for any $r \leq \theta \leq \eta < t$

$$\left\|D_\eta D_\theta \xi_t\right\|_{2p}^2 = \left\|D_\theta \xi_t D_\eta M_{\theta,t}\right\|_{2p}^2 \leq \|D_\theta \xi_t\|_{4p}^2 \left\|D_\eta M_{\theta,t}\right\|_{4p}^2$$
$$\leq C_p \left\|h''\right\|^2 \exp\left(2(4p-1)\left\|h'\right\|^2 (t-\theta)\right)(t-\eta). \tag{3.11}$$

An application of Minkowski's inequality implies that

$$\left\|\left\|D^2 \xi_t\right\|_{H\otimes H}\right\|_{2p}^2 \leq \int_r^t \int_r^t \left\|D_\eta D_\theta \xi_t\right\|_{2p}^2 d\theta d\eta.$$

This yields (3.7).

For the negative moments of $\|D\xi_t\|_H$, by Jensen's inequality we have

$$E\left(\|D\xi_t\|_H^{-2\gamma}\right) = E\left(\int_r^t |D_\theta \xi_t|^2 d\theta\right)^{-\gamma} \leq (t-r)^{-\gamma-1} \int_r^t E\,|D_\theta \xi_t|^{-2\gamma} d\theta.$$

Then, (3.8) follows immediately. □

The moment estimates of the Malliavin derivatives of the difference $\xi_t - \xi_s$ can also be obtained in a similar way. The next lemma gives these estimates.

Lemma 3.2 *For* $0 \le s < t \le T$ *and* $p \ge 1$, *we have*

$$\left\| \|D(\xi_t - \xi_s)\|_H \right\|_{2p} < C(t-s)^{\frac{1}{2}}, \tag{3.12}$$

and

$$\left\| \left\| D^2(\xi_t - \xi_s) \right\|_{H \otimes H} \right\|_{2p} < C(t-s)^{\frac{3}{2}}. \tag{3.13}$$

Proof Similar to (3.3), we have

$$\begin{aligned} D_\theta \xi_t &= D_\theta \xi_s + \mathbf{1}_{[s,t]}(\theta) - \int_{\theta \vee s}^{t} \int_{\mathbb{R}} h'(y - \xi_u) D_\theta \xi_u W(du, dy) \\ &= D_\theta \xi_s + \mathbf{1}_{[s,t]}(\theta) - I_\theta^t \left(h' D_\theta \xi_\cdot\right), \end{aligned} \tag{3.14}$$

where henceforth for any process $Y = (Y_t, 0 \le t \le T)$ and $f \in L^2(\mathbb{R})$, we denote

$$I_\theta^t(fY_\cdot) = \mathbf{1}_{[s,t]}(\theta) \int_\theta^t \int_{\mathbb{R}} f(y - \xi_u) Y_u W(du, dy).$$

Applying the Burkholder-Davis-Gundy inequality with (3.9), we obtain for $s \le \theta \le t$

$$\begin{aligned} \left\| I_\theta^t \left(h' D_\theta \xi_\cdot\right) \right\|_{2p}^2 &\le \left(E \left| \int_\theta^t \int_{\mathbb{R}} \left|h'(y - \xi_u) D_\theta \xi_u\right|^2 du dy \right|^p \right)^{\frac{1}{p}} \\ &\le \left\|h'\right\|^2 \exp\left((2p-1)\left\|h'\right\|^2 (t-\theta)\right)(t-\theta). \end{aligned} \tag{3.15}$$

Then (3.12) follows from (3.14) and (3.15) since

$$\begin{aligned} \left(E\|D\xi_t - D\xi_s\|_H^{2p}\right)^{\frac{1}{p}} &= \left[E\left(\int_0^T \left| \mathbf{1}_{[s,t]}(\theta) + I_\theta^t\left(h' D_\theta \xi_\cdot\right)\right|^2 d\theta \right)^p \right]^{\frac{1}{p}} \\ &\le \int_0^T \left(E\left| \mathbf{1}_{[s,t]}(\theta) + I_\theta^t\left(h' D_\theta \xi_\cdot\right)\right|^{2p} \right)^{\frac{1}{p}} d\theta \\ &\le 2(t-s) + 2\int_s^t \left(E\left| I_\theta^t\left(h' D_\theta \xi_\cdot\right)\right|^{2p} \right)^{\frac{1}{p}} d\theta \\ &\le 2\left(1 + \left\|h'\right\|^2 \exp\left((2p-1)\left\|h'\right\|^2 (t-s)\right)\right)(t-s). \end{aligned}$$

For moments of $D^2(\xi_t - \xi_s)$, from (3.14) we have

$$D^2_{\eta,\theta}(\xi_t - \xi_s) = -D_\eta I^t_\theta\left(h' D_\theta \xi_.\right) = I^t_\eta\left(h'' D_\theta \xi_. D_\eta \xi_.\right) - I^t_\eta\left(h' D^2_{\eta,\theta}\xi_.\right). \quad (3.16)$$

In a similar way as above we can get (3.13). □

Next we derive some estimates for the density $p^W(r, x; t, y)$ of the conditional transition probability defined in (1.8). Denote

$$u_t \equiv \frac{D\xi_t}{\|D\xi_t\|^2_H}. \quad (3.17)$$

The next two lemmas give estimates of the divergence of u_t and $u_t - u_s$, which are important to derive the moment estimates of $p^W(r, x; t, y)$.

Lemma 3.3 *For any $p \geq 1$ and $0 \leq r < t \leq T$, we have*

$$\|\delta(u_t)\|_p \leq C(t-r)^{-\frac{1}{2}}. \quad (3.18)$$

Proof Using the estimate (2.1) we obtain

$$\begin{aligned}\|\delta(u_t)\|_p &= \left(E|\delta(u_t)|^p\right)^{\frac{1}{p}} = \left[E\left(E^B|\delta(u_t)|^p\right)\right]^{\frac{1}{p}} \\ &\leq C_p\left(E\left[\left\|E^B u_t\right\|^p_H + \left(E^B\|Du_t\|^p_{H\otimes H}\right)\right]\right)^{\frac{1}{p}} \\ &\leq C_p\left(\|\|u_t\|_H\|_p + \left\|\|Du_t\|_{H\otimes H}\right\|_p\right).\end{aligned}$$

We have

$$Du_t = \frac{D^2\xi_t}{\|D\xi_t\|^2_H} - 2\frac{\langle D^2\xi_t, D\xi_t \otimes D\xi_t\rangle_{H\otimes H}}{\|D\xi_t\|^4_H},$$

and consequently $\|Du_t\|_{H\otimes H} \leq \frac{3\|D^2\xi_t\|_{H\otimes H}}{\|D\xi_t\|^2_H}$. Hence, for any positive numbers $\alpha, \beta > 1$ such that $\frac{1}{a} + \frac{1}{\beta} = \frac{1}{p}$, applying (3.7) and (3.8) we obtain (3.18):

$$\begin{aligned}\|\delta(u_t)\|_p &\leq C_p\left(\left\|\|D\xi_t\|^{-1}_H\right\|_p + 3\left\|D^2\xi_t\right\|_{L^\alpha(\Omega, H\otimes H)}\left\|\|D\xi_t\|^{-2}_H\right\|_\beta\right) \\ &\leq C\left(p, \left\|h'\right\|, \left\|h''\right\|, T\right)\left((t-r)^{-\frac{1}{2}} + (t-r)^{\frac{3}{2}}(t-r)^{-1}\right).\end{aligned}$$

This proves the lemma. □

Lemma 3.4 *For $p \geq 1$, and $0 \leq r < s < t \leq T$,*

$$\|\delta(u_t - u_s)\|_{2p} \leq C(t-s)^{\frac{1}{2}}(s-r)^{-\frac{1}{2}}(t-r)^{-\frac{1}{2}}. \quad (3.19)$$

Proof Using (3.14) we can write

$$u_t - u_s = \frac{D\xi_t}{\|D\xi_t\|_H^2} - \frac{D\xi_s}{\|D\xi_s\|_H^2} = -A_1 + A_2 - A_3,$$

where

$$A_1 = D\xi_s\left(\frac{1}{\|D\xi_s\|_H^2} - \frac{1}{\|D\xi_t\|_H^2}\right), \quad A_2 = \frac{\mathbf{1}_{[s,t]}(\theta)}{\|D\xi_t\|_H^2}, A_3 = \frac{I_\theta^t\left(h' D_\theta \xi_\cdot\right)}{\|D\xi_t\|_H^2}.$$

As a consequence, we have

$$\|\delta\left(u_t - u_s\right)\|_{2p} \le \sum_{i=1}^{3} \|\delta A_i\|_{2p}. \tag{3.20}$$

For simplicity we introduce the following notation

$$V_t \equiv \|D\xi_t\|_H, N_t \equiv \left\|D^2\xi_t\right\|_{H\otimes H}, Y_i = \left\|D^i\left(\xi_t - \xi_s\right)\right\|_{H^{\otimes i}}, \quad i = 1, 2.$$

Note that

$$\|A_1\|_H = \frac{|\langle D\xi_t - D\xi_s, D\xi_t + D\xi_s\rangle|}{\|D\xi_s\|_H \|D\xi_t\|_H^2} \le Y_1\left(V_t^{-2} + V_s^{-1}V_t^{-1}\right),$$

and

$$\begin{aligned}
\|DA_1\|_{H\otimes H} &= \left\|D\left(\frac{D\xi_s\langle D\xi_t - D\xi_s, D\xi_t + D\xi_s\rangle}{\|D\xi_s\|_H^2 \|D\xi_t\|_H^2}\right)\right\|_{H\otimes H} \\
&\le Y_1 N_s\left(V_s^{-2}V_t^{-1} + V_s^{-1}V_t^{-2}\right) + Y_2\left(V_s^{-1}V_t^{-1} + V_t^{-2}\right) \\
&\quad + Y_1\left(N_t + N_s\right)V_s^{-1}V_t^{-2} \\
&\quad + 2Y_1\left[N_s\left(V_s^{-2}V_t^{-1} + V_s^{-1}V_t^{-2}\right) + N_t\left(V_t^{-3} + V_s^{-1}V_t^{-2}\right)\right].
\end{aligned}$$

As a consequence, applying Lemma 2.1 and Hölder's inequality we get

$$\begin{aligned}
\|\delta\left(A_1\right)\|_{2p} &\le C\left(\|\|A_1\|_H\|_{2p} + \left\|\|DA_1\|_{H\otimes H}\right\|_{2p}\right) \\
&\le C\|Y_1\|_{4p}\left(\left\|V_t^{-2}\right\|_{4p} + \left\|V_t^{-1}\right\|_{8p}\left\|V_s^{-1}\right\|_{8p}\right) \\
&\quad + C\|Y_1\|_{8p}\|N_s\|_{8p}\left(\left\|V_t^{-1}\right\|_{8p}\left\|V_s^{-2}\right\|_{8p} + \left\|V_s^{-1}\right\|_{8p}\left\|V_t^{-2}\right\|_{8p}\right) \\
&\quad + C\|Y_2\|_{4p}\left(\left\|V_t^{-1}\right\|_{8p}\left\|V_s^{-1}\right\|_{8p} + \left\|V_t^{-2}\right\|_{4p}\right)
\end{aligned}$$

$$+C\|Y_1\|_{8p}\left(\|N_s\|_{8p}+\|N_t\|_{8p}\right)\left\|V_t^{-2}\right\|_{8p}\left\|V_s^{-1}\right\|_{8p}$$
$$+2C\|Y_1\|_{8p}\|N_s\|_{8p}\left(\left\|V_t^{-1}\right\|_{8p}\left\|V_s^{-2}\right\|_{8p}+\left\|V_t^{-2}\right\|_{8p}\left\|V_s^{-1}\right\|_{8p}\right)$$
$$+2C\|Y_1\|_{8p}\|N_t\|_{8p}\left(\left\|V_t^{-3}\right\|_{4p}+\left\|V_t^{-2}\right\|_{8p}\left\|V_s^{-1}\right\|_{8p}\right).$$

From Lemma 3.1 and Lemma 3.2 it follows that

$$\|\delta(A_1)\|_{2p}\le C(t-s)^{\frac{1}{2}}(s-r)^{-\frac{1}{2}}(t-r)^{-\frac{1}{2}}. \tag{3.21}$$

Note that $\|A_2\|_H=\left\|\frac{\mathbf{1}_{[s,t]}(\theta)}{\|D\xi_t\|_H^2}\right\|_H=\|D\xi_t\|_H^{-2}(t-s)^{\frac{1}{2}}$ and

$$\|DA_2\|_{H\otimes H}\le 2\|D\xi_t\|_H^{-3}\left\|D^2\xi_t\right\|_{H\otimes H}(t-s)^{\frac{1}{2}}.$$

Then, by Lemma 2.1, Hölder's inequality and Lemma 3.1 we see that

$$\begin{aligned}\|\delta(A_2)\|_{2p}&\le C\left(\|\|A_2\|_H\|_{2p}+\|DA_2\|_{2p}\right)\\&\le C(t-s)^{\frac{1}{2}}\left(\left\|V_t^{-2}\right\|_{2p}+\left\|D^2\xi_t\right\|_{4p}\left\|V_t^{-1}\right\|_{4p}\right)\\&\le 2C(t-s)^{\frac{1}{2}}\left((t-r)^{-1}+1\right).\end{aligned}\tag{3.22}$$

For the term A_3, we apply Minkowski's inequality and the Burkholder-Davis-Gundy inequality and use (3.15). Thus for any $p\ge 1$,

$$\begin{aligned}\left\|\left|\left|I_\theta^t\left(h'D_\theta\xi.\right)\right|\right|_H\right\|_{2p}&=\left(E\left|\int_s^t I_\theta^t(h'D_\theta\xi.)^2d\theta\right|^p\right)^{\frac{1}{2p}}\\&\le C_p\left(\int_s^t\left\|I_\theta^t\left(h'D\xi.\right)\right\|_{2p}^2d\theta\right)^{\frac{1}{2}}\\&\le C_p\left\|h'\right\|\exp\left((2p-1)\left\|h'\right\|^2(t-r)\right)(t-s)^{\frac{1}{2}}.\end{aligned}\tag{3.23}$$

From (3.16) it follows that

$$\begin{aligned}\|DA_3\|_{H\otimes H}\le&\left\|D^2(\xi_t-\xi_s)\right\|_{H\otimes H}\|D\xi_t\|_H^{-2}\\&+2\left|\left|I_\theta^t\left(h'D\xi.\right)\right|\right|_H\left\|D^2\xi_t\right\|_{H\otimes H}\|D\xi_t\|_H^{-3}.\end{aligned}$$

Combining this with Lemma 2.1, Hölder's inequality, Lemma 3.2 and (3.23) we deduce

$$\begin{aligned}\|\delta(A_3)\|_{2p} &\le C_p\left(\|\|A_3\|_H\|_{2p}+\|DA_3\|_{2p}\right)\\ &\le C_p\left\|\left|\left|I_s^t\left(h' D_\theta \xi_\cdot\right)\right|\right|_H\right\|_{2p}\left\|V_t^{-2}\right\|_{2p}+\|Y_2\|_{4p}\left\|V_t^{-2}\right\|_{4p}\\ &\quad +2C_p\left\|\left|\left|I_s^t\left(h' D_\theta \xi_\cdot\right)\right|\right|_H\right\|_{4p}\left\|V_t^{-3}\right\|_{8p}\|N_t\|_{8p}\\ &\le C\,(t-s)^{\frac{1}{2}}\,(t-r)^{-1}\,. \end{aligned} \tag{3.24}$$

Substituting (3.21), (3.22) and (3.24) into (3.20) yields (3.19). □

Now we provide the moment estimates for the conditional transition probability density $p^W(r,x;t,y)$.

Lemma 3.5 *Let $c=1\vee\|h\|^2$. For any $0\le r<t\le T$, $y\in\mathbb{R}$ and $p\ge 1$,*

$$\left(E\left|p^W(r,x;t,y)\right|^{2p}\right)^{\frac{1}{2p}}\le 2\exp\left(-\frac{(x-y)^2}{64pc\,(t-r)}\right)\|\delta(u_t)\|_{4p}\,. \tag{3.25}$$

Proof By Lemma 2.2 we can write

$$p^W(r,x;t,y)=E^B\left(\mathbf{1}_{\{\xi_t>y\}}\delta(u_t)\right)=E^B[\mathbf{1}_{\{B_r^t+I_r^t(h)>y-x\}}\delta(u_t)], \tag{3.26}$$

where B_r^t and $I_r^t(h)$ are defined in (3.2). Then, (2.2) implies

$$\begin{aligned}\left(E\left|p^W(r,x;t,y)\right|^{2p}\right)^{\frac{1}{2p}} &\le\left(E\left[\left(P^B\left(\left|B_r^t+I_r^t(h)\right|\ge|y-x|\right)\right)^p\left(E^B\left|\delta(u_t)\right|^2\right)^p\right]\right)^{\frac{1}{2p}}\\ &\le\|\delta(u_t)\|_{4p}\left(E\left(P^B\left(\left|B_r^t+I_r^t(h)\right|\ge|y-x|\right)\right)^{2p}\right)^{\frac{1}{4p}}.\end{aligned} \tag{3.27}$$

Applying Chebyshev and Jensen's inequalities, we have for $p\ge 1$,

$$\begin{aligned}E\left|P^B\left(\left|B_r^t+I_r^t(h)\right|\ge|y-x|\right)\right|^{2p} &\le\exp\left(\frac{-2p\,(x-y)^2}{32pc\,(t-r)}\right)E\left|E^B\exp\frac{\left(B_r^t+I_r^t(h)\right)^2}{32pc\,(t-r)}\right|^{2p}\\ &\le\exp\left(\frac{-(x-y)^2}{16c\,(t-r)}\right)E\exp\frac{\left(B_r^t+I_r^t(h)\right)^2}{16c\,(t-r)}.\end{aligned} \tag{3.28}$$

Using the fact that for $0\le\nu<1/8$ and Gaussian random variables X,Y,

$$Ee^{\nu(X+Y)^2}\le Ee^{2\nu\left(X^2+Y^2\right)}\le(Ee^{4\nu X^2})^{\frac{1}{2}}(Ee^{4\nu Y^2})^{\frac{1}{2}}=(1-8\nu)^{-\frac{1}{2}}\,,$$

and noticing that B_r^t and $I_r^t(h)$ are Gaussian, we have

$$E \exp \frac{\left(B_r^t + I_r^t(h)\right)^2}{16c\,(t-r)} \le \left(1 - \frac{1}{2c}\right)^{-\frac{1}{2}} \le \sqrt{2}. \tag{3.29}$$

Combining (3.27)–(3.29), we get (3.25). □

4 Hölder continuity in spatial variable

In this section, we obtain the Hölder continuity of $X_t(y)$ with respect to y. More precisely, we show that for $t > 0$ fixed, $X_t(y)$ is almost surely Hölder continuous in y with any exponent in $(0, 1/2)$. This result was proved in [4]. Here we provide a different proof based on Malliavin calculus. We continue to use the notations B_r^t, $I_r^t(h)$ defined by (3.2) and u_t defined by (3.17).

Proposition 4.1 *Suppose that $h \in H_2^2(\mathbb{R})$ and $X_0 = \mu \in L^2(\mathbb{R})$ is bounded. Then, for any $p > 1$, $\alpha \in (0,1)$ and $T > 0$, there exists a constant C depending only on α, p, T, $\|h\|_{2,2}$ and $\|\mu\|_{L^2(\mathbb{R})}$ such that for any $t \in (0, T]$, $y_1, y_2 \in \mathbb{R}$,*

$$E\left|X_t\left(y_2\right) - X_t\left(y_1\right)\right|^{2p} \le C(1 + t^{-p})\left|y_2 - y_1\right|^{\alpha p}. \tag{4.1}$$

Moreover, the term t^{-p} can be replaced by $\|\mu\|_\lambda$ if we also assume that μ is Hölder continuous with exponent $\lambda > \frac{1}{2}$.

Proof Suppose $y_1 < y_2$. We will use the convolution representation (1.9), where the two terms $X_{t,1}(y)$ and $X_{t,2}(y)$ will be estimated separately.

We start with $X_{t,2}(y)$. Note that $\mathbf{1}_{\{\xi_t > y_1\}} - \mathbf{1}_{\{\xi_t > y_2\}} = \mathbf{1}_{\{y_1 < \xi_t \le y_2\}}$ and

$$E^B \mathbf{1}_{\{y_1 < \xi_t \le y_2\}} = P^B\left\{y_1 < \xi_t \le y_2\right\} = \int_{y_1}^{y_2} p^W(r, x; t, z)\, dz.$$

Therefore by (3.26) and the Cauchy-Schwartz inequality we have

$$\begin{aligned}\left|p^W(r, x; t, y_1) - p^W(r, x; t, y_2)\right|^2 &= \left|E^B\left[\mathbf{1}_{\{y_1 < \xi_t < y_2\}} \delta(u_t)\right]\right|^2 \\ &\le E^B \left|\delta(u_t)\right|^2 \int_{y_1}^{y_2} p^W(r, x; t, z)\, dz.\end{aligned}$$

Then, applying Jensen, Hölder and Minkowski's inequalities we have

$$\begin{aligned}&\left(E\left|p^W(r,x;t,y_1)-p^W(r,x;t,y_2)\right|^{2(2p-1)}\right)^{\frac{1}{2p-1}}\\&\le\left(EE^B\left|\delta(u_t)\right|^{2(2p-1)}\left|\int_{y_1}^{y_2}p^W(r,x;t,z)\,dz\right|^{(2p-1)}\right)^{\frac{1}{2p-1}}\\&\le\|\delta(u_t)\|^2_{4(2p-1)}\int_{y_1}^{y_2}\left\|p^W(r,x;t,z)\right\|_{2(2p-1)}dz.\end{aligned}\tag{4.2}$$

Lemma 3.3 and Lemma 3.5 yield

$$\begin{aligned}&\int_{\mathbb{R}}\left(E\left|p^W(r,x;t,y_1)-p^W(r,x;t,y_2)\right|^{2(2p-1)}\right)^{\frac{1}{2p-1}}dx\\&\le C\int_{\mathbb{R}}\|\delta(u_t)\|^3_{4(2p-1)}\int_{y_1}^{y_2}\exp\left(\frac{-(z-x)^2}{32(2p-1)c(t-r)}\right)dzdx\\&\le C(t-r)^{-1}(y_2-y_1).\end{aligned}\tag{4.3}$$

On the other hand, the left hand side of (4.3) can be estimated differently again by using Lemma 3.5:

$$\begin{aligned}&\int_{\mathbb{R}}\left(E\left|p^W(r,x;t,y_1)-p^W(r,x;t,y_2)\right|^{2(2p-1)}\right)^{\frac{1}{2p-1}}dx\\&\le 2\int_{\mathbb{R}}\Sigma_{i=1,2}\left(E\left|p^W(r,x;t,y_i)\right|^{2(2p-1)}\right)^{\frac{1}{2p-1}}dx\\&\le C_p\int_{\mathbb{R}}\Sigma_{i=1,2}\|\delta(u_t)\|^2_{4(2p-1)}\exp\left(\frac{-(y_i-x)^2}{64pc(t-r)}\right)dx\le C(t-r)^{-\frac{1}{2}}.\end{aligned}\tag{4.4}$$

Then (4.3) and (4.4) yield that for any $\alpha,\beta>0$ with $\alpha+\beta=1$

$$\begin{aligned}&\int_{\mathbb{R}}\left(E\left|p^W(r,x;t,y_1)-p^W(r,x;t,y_2)\right|^{2(2p-1)}\right)^{\frac{1}{2p-1}}dx\\&\le C(t-r)^{-\alpha-\frac{1}{2}\beta}(y_2-y_1)^{\alpha}.\end{aligned}\tag{4.5}$$

Since μ is bounded, it follows from [4, Lemma 4.1] that

$$E\left|\int_0^t\int_{\mathbb{R}}\left(p^W(r,x;t,y_2)-p^W(r,x;s,y_1)\right)^2 Z(drdx)\right|^{2p}$$
$$\leq C\left(E\left|\int_0^t\int_{\mathbb{R}}\left(p^W(r,x;t,y_2)-p^W(r,x;s,y_1)\right)^2 drdx\right|^{2p-1}\right)^{\frac{p}{2p-1}}, \quad (4.6)$$

for any $p\geq 1, 0\leq s\leq t\leq T$ and $y_1,y_2\in\mathbb{R}$. Then, applying Minkowski's inequality we obtain for any $0<\alpha<1$,

$$\left(E\left|X_{t,2}(y_2)-X_{t,2}(y_1)\right|^{2p}\right)^{\frac{1}{p}}$$
$$\leq\int_0^t\int_{\mathbb{R}}\left(E\left|p^W(r,x;t,y_1)-p^W(r,x;t,y_2)\right|^{2(2p-1)}\right)^{\frac{1}{2p-1}} dxdr$$
$$\leq C\int_0^t (t-r)^{-\alpha-\frac{1}{2}\beta}(y_2-y_1)^\alpha dr\leq C(y_2-y_1)^\alpha \quad (4.7)$$

since $(t-r)^{-\alpha-\frac{1}{2}\beta}=(t-r)^{-(1+\alpha)/2}$ is integrable for all $0<\alpha<1$.

Now we consider $X_{t,1}(y)$ in (1.9). Applying Minkowski's inequality and using (4.2) with $2p-1$ replaced by p we get

$$E\left|X_{t,1}(y_2)-X_{t,1}(y_1)\right|^{2p}\leq\left(\int_{\mathbb{R}}\left(E\left|p(0,x;t,y_1)-p^W(0,x;t,y_2)\right|^{2p}\right)^{\frac{1}{2p}}\mu(x)dx\right)^{2p}$$
$$\leq C\left\{\int_{\mathbb{R}}\left(\int_{y_1}^{y_2}\left\|p^W(0,x;t,z)\right\|_{2p}dz\right)^{1/2}\|\delta(u_t)\|_{4p}\mu(x)dx\right\}^{2p}$$
$$\leq C\|\delta(u_t)\|_{4p}^{2p}\|\mu\|_{L^2(\mathbb{R})}^{2p}\left(\int_{\mathbb{R}}\int_{y_1}^{y_2}\exp\left(-\frac{(z-x)^2}{64pct}\right)dzdx\right)^p$$
$$\leq C\|\mu\|_{L^2(\mathbb{R})}^{2p}t^{-p}(y_2-y_1)^p.$$

Combining the above estimate with (4.7) we get (4.1).

If in addition to that $\mu\in L^2(\mathbb{R})$, we assume μ is Hölder continuous, then

$$E\left|X_{t,1}(y_2)-X_{t,1}(y_1)\right|^{2p}$$
$$=E\left|E^B\mu(y_2-B_r^t-I_r^t(h))-E^B\mu(y_1-B_r^t-I_r^t(h))\right|^{2p}$$

$$\leq EE^B \left|\mu(y_2 - B_r^t - I_r^t(h)) - \mu(y_1 - B_r^t - I_r^t(h))\right|^{2p}$$
$$\leq \|u\|_\lambda (y_2 - y_1)^{2\lambda p},$$

where in the first equality we used the fact that

$$p^W(r, x; t, y) = E^B \delta_y(x + B_r^t + I_r^t) = E^B \delta_x(y - B_r^t - I_r^t),$$

where δ_x is the Dirac function at x. Combining the above inequality with (4.7) we complete the proof. □

5 Hölder continuity in time variable

In this section we show that for any fixed $y \in \mathbb{R}$, $X_t(y)$ is Hölder continuous in t with any exponent in $(0, 1/4)$.

Proposition 5.1 *Suppose that $h \in H_2^2(\mathbb{R})$ and $X_0 = \mu \in L^2(\mathbb{R})$ is bounded. Then, for any $p \geq 1$, $y \in \mathbb{R}$ and $0 \leq s < t \leq T$,*

$$E\,|X_t(y) - X_s(y)|^{2p} \leq Ct^{-p}(t - s)^{\frac{p}{2} - \frac{1}{4}}, \tag{5.1}$$

where the constant C depends only on p, T, $\|h\|_{2,2}$ and $\|\mu\|_{L^2(\mathbb{R})}$. Moreover, if μ is also Hölder continuous with exponent $\lambda > \frac{1}{2}$, then

$$E\,|X_t(y) - X_s(y)|^{2p} \leq C(1 + \|\mu\|_\lambda)(t - s)^{\frac{p}{2} - \frac{1}{4}}. \tag{5.2}$$

Suppose $\mu \in L^2(\mathbb{R})$ is bounded. Recall that we write $X_t(y) = X_{t,1}(y) + X_{t,2}(y)$ in (1.9). We start by estimating $X_{\cdot,2}(y)$. We write

$$X_{t,2}(y) - X_{s,2}(y) = \int_0^s \int_{\mathbb{R}} \left(p^W(r, x; t, y) - p^W(r, x; s, y)\right) Z(drdx) + \int_s^t \int_{\mathbb{R}} p^W(r, x; t, y)\, Z(drdx). \tag{5.3}$$

We are going to estimate the two terms separately.

Lemma 5.2 *For any $0 \leq s < t \leq T$, $y \in \mathbb{R}$ and $p \geq 1$, we have*

$$E\left(\int_0^s \int_{\mathbb{R}} \left(p^W(r, x; t, y) - p^W(r, x; s, y)\right) Z(drdx)\right)^{2p} \leq C(t - s)^{\frac{p}{2} - \frac{1}{4}}. \tag{5.4}$$

Proof From (3.26), we have for $0 < r < s < t \leq T$,

$$
\begin{aligned}
p^W(r,x;t,y) - p^W(r,x;s,y) &= E^B\left[\mathbf{1}_{\{\xi_t>y\}}\delta(u_t) - \mathbf{1}_{\{\xi_s>y\}}\delta(u_s)\right] \\
&= E^B\left[\left(1_{\{\xi_t>y\}} - \mathbf{1}_{\{\xi_s>y\}}\right)\delta(u_t) + \mathbf{1}_{\{\xi_s>y\}}\delta(u_t-u_s)\right],
\end{aligned}
$$

Let $I_1 \equiv \left(1_{\{\xi_t>y\}} - \mathbf{1}_{\{\xi_s>y\}}\right)\delta(u_t)$ and $I_2 \equiv \mathbf{1}_{\{\xi_s>y\}}\delta(u_t-u_s)$. Then (4.6) implies

$$
\begin{aligned}
&E\left(\int_0^s\int_{\mathbb{R}}\left(p^W(r,x;t,y) - p^W(r,x;s,y)\right)Z(drdx)\right)^{2p} \\
&\leq \left[E\left(\int_0^s\int_{\mathbb{R}}\left(E^B[I_1+I_2]\right)^2 drdx\right)^{2p-1}\right]^{\frac{p}{2p-1}} \\
&\leq C\sum_{i=1,2}\left[E\left(\int_0^s\int_{\mathbb{R}}\left(E^B I_i\right)^2 drdx\right)^{2p-1}\right]^{\frac{p}{2p-1}}. \qquad (5.5)
\end{aligned}
$$

First, we study the term I_1. Note that

$$
\left(1_{\{\xi_t>y\}} - \mathbf{1}_{\{\xi_s>y\}}\right)^2 = 1_{\{\xi_s\leq y<\xi_t\}} + \mathbf{1}_{\{\xi_t\leq y<\xi_s\}} =: A_1 + A_2.
$$

Then we can write

$$
\begin{aligned}
&\left[E\left(\int_0^s\int_{\mathbb{R}} E^B I_1^2 drdx\right)^{2p-1}\right]^{\frac{1}{2p-1}} \\
&= \left[E\left(\int_0^s\int_{\mathbb{R}} E^B\left[(A_1+A_2)\delta(u_t)\right]^2 drdx\right)^{2p-1}\right]^{\frac{1}{2p-1}} \\
&\leq 2\sum_{i=1,2}\left[E\left(\int_0^s\int_{\mathbb{R}} E^B\left[A_i\delta(u_t)\right]^2 drdx\right)^{2p-1}\right]^{\frac{1}{2p-1}}. \qquad (5.6)
\end{aligned}
$$

Applying Minkowski, Jensen and Hölder's inequalities we deduce that for $i = 1, 2$ and for any conjugate pair (p_1, q_1)

$$\left[E\left(\int_0^s\int_{\mathbb{R}} E^B\left[A_i\delta\left(u_t\right)\right]^2 drdx\right)^{2p-1}\right]^{\frac{1}{2p-1}}$$
$$\le \int_0^s \left(E\left|\int_{\mathbb{R}} A_i\left|\delta\left(u_t\right)\right|^2 dx\right|^{2p-1}\right)^{\frac{1}{2p-1}} dr$$
$$\le \int_0^s \left\|\left(\int_{\mathbb{R}} A_i dx\right)^{\frac{1}{p_1}}\left(\int_{\mathbb{R}} A_i\left|\delta\left(u_t\right)\right|^{2q_1} dx\right)^{\frac{1}{q_1}}\right\|_{2p-1} dr$$
$$\le \int_0^s \left\|\left|\int_{\mathbb{R}} A_i dx\right|^{\frac{1}{p_1}}\right\|_{2(2p-1)}\left\|\left(\int_{\mathbb{R}} A_i\left|\delta\left(u_t\right)\right|^{2q_1} dx\right)^{\frac{1}{q_1}}\right\|_{2(2p-1)} dr. \quad (5.7)$$

Notice that

$$\{\xi_s \le y < \xi_t\} = \left\{y - B_r^t - I_r^t(h) < x \le y - B_r^s - I_r^s(h)\right\},$$
$$\{\xi_t \le y < \xi_s\} = \left\{y - B_r^s - I_r^s(h) < x \le y - B_r^t - I_r^t(h)\right\}.$$

Then, for $i = 1, 2$, we have

$$\left|\int_{\mathbb{R}} A_i dx\right| = \left|B_s^t + I_s^t(h)\right|.$$

Hence for $p_1 = 1 - \frac{1}{2p}$,

$$\left\|\left|\int_{\mathbb{R}} A_i dx\right|^{\frac{1}{p_1}}\right\|_{2(2p-1)} \le C\left(t-s\right)^{\frac{1}{2}-\frac{1}{4p}}. \quad (5.8)$$

On the other hand, we have

$$\{\xi_s \le y < \xi_t\} = \left\{B_r^s + I_r^s(h) \le y - x < B_r^t + I_r^t(h)\right\}$$
$$\subset \left\{|x-y| \le \left|B_r^t + I_r^t(h)\right| + \left|B_r^s + I_r^s(h)\right|\right\}.$$

Similarly

$$\{\xi_t \le y < \xi_s\} \subset \left\{|x-y| \le \left|B_r^t + I_r^t(h)\right| + \left|B_r^s + I_r^s(h)\right|\right\}.$$

Applying Chebyshev's inequality and (3.29), we deduce that for $i = 1, 2$,

$$\begin{aligned} E\left(A_i\right) &\leq EP^B\left\{|x-y| \leq \left|B_r^t + I_r^t(h)\right| + \left|B_r^s + I_r^s(h)\right|\right\} \\ &\leq \exp\left(\frac{-(x-y)^2}{32c\,(t-r)}\right) E\exp\left(\frac{\left|B_r^t + I_r^t(h)\right|^2}{16c(t-r)} + \frac{\left|B_r^s + I_r^s(h)\right|^2}{16c(s-r)}\right) \\ &\leq 2\exp\left(-\frac{(x-y)^2}{32c\,(t-r)}\right). \end{aligned} \tag{5.9}$$

Using Minkowski and Hölder's inequalities, from (5.9) and Lemma 3.3 we obtain that for $q_1 = 2p \leq 2\,(2p-1)$,

$$\begin{aligned} \left\| \left(\int_{\mathbb{R}} A_i \left|\delta\left(u_t\right)\right|^{2q_1} dx \right)^{\frac{1}{q_1}} \right\|_{2(2p-1)} &\leq \left(\int_{\mathbb{R}} \left\| A_i \left|\delta\left(u_t\right)\right|^{2q_1} \right\|_{\frac{2(2p-1)}{q_1}} dx \right)^{\frac{1}{q_1}} \\ &\leq \left(\int_{\mathbb{R}} \left(EA_i\right)^{\frac{q_1}{4(2p-1)}} \left\|\delta\left(u_t\right)\right\|_{8(2p-1)}^{2q_1} dx \right)^{\frac{1}{q_1}} \\ &\leq C\,(t-r)^{\frac{1}{4p}-1}. \end{aligned} \tag{5.10}$$

Substituting (5.8) and (5.10) into (5.7) we obtain

$$\begin{aligned} \left[E\left(\int_0^s \int_{\mathbb{R}} E^B\left[A_i \delta\left(u_t\right)\right]^2 dr dx \right)^{2p-1} \right]^{\frac{1}{2p-1}} &\leq C\,(t-s)^{\frac{1}{2}-\frac{1}{4p}} \int_0^s (t-r)^{\frac{1}{4p}-1} dr \\ &\leq C\,(t-s)^{\frac{1}{2}-\frac{1}{4p}}. \end{aligned} \tag{5.11}$$

Combining (5.6) and (5.11), we have

$$\left[E\left(\int_0^s \int_{\mathbb{R}} E^B I_1^2 dr dx \right)^{2p-1} \right]^{\frac{1}{2p-1}} \leq C\,(t-s)^{\frac{1}{2}-\frac{1}{4p}}. \tag{5.12}$$

We turn into the term I_2. From Lemma 2.3 we can deduce as in Lemma 3.5 that

$$\left(E\left(E^B I_2\right)^{2(2p-1)}\right)^{\frac{1}{2p-1}} \leq 2\exp\left(\frac{-(x-y)^2}{32\,(2p-1)\,c\,(s-r)}\right) \left\|\delta\left(u_t - u_s\right)\right\|_{4(2p-1)}^2.$$

Then applying Minkowski's inequality and Lemma 3.4, we obtain

$$\left[E\left(\int_0^s\int_{\mathbb{R}}\left(E^B I_2\right)^2 drdx\right)^{2p-1}\right]^{\frac{1}{2p-1}} \le \int_0^s\int_{\mathbb{R}}\left(E\left(E^B I_2\right)^{2(2p-1)}\right)^{\frac{1}{2p-1}} drdx$$
$$\le 2\int_0^s\int_{\mathbb{R}}\exp\left(-\frac{(x-y)^2}{32\,(2p-1)\,(s-r)}\right)\|\delta\,(u_t-u_s)\|^2_{4(2p-1)}\,dxdr$$
$$\le C\,(t-s)\int_0^s (s-r)^{\frac{1}{2}-1}\,(t-r)^{-1}\,dr \le C\,(t-s)^{\frac{1}{2}-\frac{1}{4p}}\,, \tag{5.13}$$

where in the last step we used that $(t-r)^{-1} \le (t-s)^{-\frac{1}{2}-\varepsilon}\,(s-r)^{-\frac{1}{2}+\varepsilon}$ for any $\varepsilon > 0$.

Substituting (5.12) and (5.13) in (5.5) we obtain (5.4). □

Lemma 5.3 *For any* $0 \le s < t \le T$ *and any* $y \in \mathbb{R}$ *and* $p \ge 1$*, we have*

$$E\left|\int_s^t\int_{\mathbb{R}} p^W\,(r,x;t,y)\,Z\,(drdx)\right|^{2p} \le C\,(t-s)^{\frac{p}{2}}\,. \tag{5.14}$$

Proof Since μ is bounded, it follows from [4, Lemma 4.1] that

$$E\left|\int_s^t\int_{\mathbb{R}} p^W\,(r,x;t,y)^2\,Z\,(drdx)\right|^{2p} \le C\left(E\left|\int_s^t\int_{\mathbb{R}} p^W\,(r,x;t,y)^2\,drdx\right|^{2p-1}\right)^{\frac{p}{2p-1}}$$

for any $p \ge 1$ and $y \in \mathbb{R}$. Applying Minkowski's inequality, Lemma 3.5 and Lemma 3.3 we obtain

$$\left[E\left(\int_s^t\int_{\mathbb{R}}\left|p^W\,(r,x;t,y)\right|^2 drdx\right)^{2p-1}\right]^{\frac{1}{2p-1}}$$
$$\le C\int_s^t\int_{\mathbb{R}}\left(E\left|p^W\,(r,x;t,y)\right|^{2(2p-1)}\right)^{\frac{1}{2p-1}} drdx$$
$$\le C\int_s^t\int_{\mathbb{R}}\exp\left(-\frac{(x-y)^2}{32c\,(t-r)}\right)\|\delta\,(u_t)\|^2_{4(2p-1)}\,drdx$$
$$\le C\int_s^t (t-r)^{\frac{1}{2}-1}\,dr \le C\,(t-s)^{\frac{1}{2}}\,.$$

Then (5.14) follows immediately. □

In summary of the above two lemmas, we get

Proposition 5.4 *For any* $p \geq 1$, $0 \leq s < t \leq T$ *and* $y \in \mathbb{R}$, *we have*

$$E\left|X_{t,2}(y) - X_{s,2}(y)\right|^{2p} \leq C(t-s)^{\frac{p}{2}-\frac{1}{4}}. \tag{5.15}$$

Now we consider $X_{t,1}(y)$. Note that

$$\begin{aligned} E\left|X_{t,1}(y) - X_{s,1}(y)\right|^{2p} &= E\left|\int_{\mathbb{R}} \left(p^W(0,z;t,y) - p^W(0,z;t,y)\right)\mu(z)dz\right|^{2p} \\ &= E\left|\int_{\mathbb{R}} \left(E^B\left[\mathbf{1}_{\{\xi_t>y\}}\delta(u_t) - \mathbf{1}_{\{\xi_s>y\}}\delta(u_s)\right]\right)\mu(z)dz\right|^{2p}. \end{aligned}$$

Then, similar to the proof for $X_{\cdot,2}(y)$ we get estimates for $X_{\cdot,1}(y)$.

Proposition 5.5 *Suppose* $X_0 = \mu \in L^2(\mathbb{R})$ *is bounded. Then, for any* $p \geq 1$, $0 \leq s < t \leq T$ *and* $y \in \mathbb{R}$, *we have*

$$E\left|X_{t,1}(y) - X_{s,1}(y)\right|^{2p} \leq Ct^{-p}(t-s)^{\frac{1}{2}p}. \tag{5.16}$$

Proof Let $I_1 \equiv \left(1_{\{\xi_t>y\}} - \mathbf{1}_{\{\xi_s>y\}}\right)\delta(u_t)$ and $I_2 \equiv \mathbf{1}_{\{\xi_s>y\}}\delta(u_t - u_s)$. Then,

$$E\left|X_{t,1}(y) - X_{s,1}(y)\right|^{2p} = E\left|\int_{\mathbb{R}} \mu(x)E^B[I_1 + I_2]dx\right|^{2p}.$$

Noticing that $\left|1_{\{\xi_t>y\}} - \mathbf{1}_{\{\xi_s>y\}}\right| = 1_{\{\xi_s\leq y<\xi_t\}} + \mathbf{1}_{\{\xi_t\leq y<\xi_s\}} =: A_1 + A_2$, and applying Fubini's theorem, Jensen, Hölder and Minkowski's inequalities, we obtain

$$\begin{aligned} E\left|\int_{\mathbb{R}} \mu(x)E^B|I_1|\,dx\right|^{2p} &\leq \sum_{i=1,2} E\left|\int_{\mathbb{R}} \mu(x)E^B[A_i\delta(u_t)]\,dx\right|^{2p} \\ &\leq \sum_{i=1,2} E[\left(\int_{\mathbb{R}} |\mu(x)\delta(u_t)|^2\,dx\right)^p \left|\int_{\mathbb{R}} A_i dx\right|^p] \\ &\leq \sum_{i=1,2} \left(\int_{\mathbb{R}} |\mu(x)|^2 \|\delta(u_t)\|_{4p}^2\,dx\right)^p \left(E\left|B_s^t + I_s^t(h)\right|^{2p}\right)^{\frac{1}{2}} \\ &\leq C\left(1 + \|h\|^2\right)\|\mu\|_{L^2}^{2p} t^{-p}(t-s)^{\frac{1}{2}p}, \end{aligned}$$

where in the last inequality we have applied (3.18) with $r = 0$.

For the term I_2, using Minkowski's inequality, (3.25) and (3.19) with $r = 0$ we have

$$E\left|\int_{\mathbb{R}} \mu(x)E^B\,|I_2|\,dx\right|^{2p} = \left(\int_{\mathbb{R}} |\mu(x)| \left(E\left|E^B \mathbf{1}_{\{\xi_s>y\}}\delta\,(u_t-u_s)\right|^{2p}\right)^{\frac{1}{2p}} dx\right)^{2p}$$
$$\le C\,\|\mu\|_\infty^{2p}\left(\int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{32cs}\right)\|\delta\,(u_t-u_s)\|_{4p}\,dx\right)^{2p}$$
$$\le C\,\|\mu\|_\infty^{2p}\,t^{-p}\,(t-s)^p\,.$$

Then we can conclude (5.16). □

Proof of Proposition 5.1 If $\mu \in L^2(\mathbb{R})$ is bounded, then (5.1) follows from Proposition 5.4 and Proposition 5.5. Now assume that μ is Hölder continuous. Then the left hand side of (5.16) can be estimated as follows

$$E\left|X_{t,1}(y) - X_{s,1}(y)\right|^{2p} = E\left|E^B\mu(y-B_0^t-I_0^t) - E^B\mu(y-B_0^s-I_0^s)\right|^{2p}$$
$$\le \|u\|_\lambda\, EE^B\left(\left|B_s^t + I_s^t\right|^{2\lambda p}\right).$$

Combining the above estimate with (5.15) we obtain (5.2). □

Proof of Theorem 1.1 It follows from Proposition 4.1 and Proposition 5.1. □

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