[Higham 01] : elementary intro. on simulation

[KP92] : our reference

Numerical Approximation of SDE

$$dX_t = a(X_t)dt + b(X_t)dW_t \ ; \quad X_0 = x_0 \qquad (1)$$

- Euler-Maruyama ; Milstein ; $\Big\rangle$ Ito-Taylor expansion
- Strong / Weak Appr. ; order of convergence.
- stability

1. Time discrete Approximations.

$$X_t = x_0 + \int_0^t a(X_s)\,ds + \int_0^t b(X_s)\,dW_s \ ; \qquad t \in [0,T].$$

consider discrete times : $0 = t_0 < t_1 < \cdots < t_N = T$, $\quad t_i = i\frac{T}{N}$.

EM: $\quad Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n \ ; \quad Y_0 = x_0 \ ; \quad n = 0, 1, \cdots, N-1.$

$$\Delta W_n = W_{t_{n+1}} - W_{t_n} \sim \mathcal{N}(0, \Delta t) \ ; \ \text{i.i.d.}$$

Milstein: $\quad Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n + \frac{1}{2} b b' ((\Delta W_n)^2 - \Delta t).$

- ($\Delta W_n \sim \sqrt{\Delta t}\,\mathcal{N}(0,1)$ i.i.d., generated by a pseudo-random number generator.)

Question:

① How do we get them? What is a "valid" scheme?

② order of convergence?

③ Higher-order schemes?

④ High-D version?

⑤ stability?

2. Ito–Taylor expansion (Wagner–Platen formula) (similar to numerical ODE)

Apply Ito's formula [chain rule for $f \in C^2$: $df(X_t) = f'(X_t)\left[a(X_t)dt + b(X_t)dW_t\right] + \frac{1}{2} f''(X_t) b^2 dt$]

$$X_t = x_0 + \int_0^t a(X_s)\,ds + \int_0^t b(X_s)\,dW_s$$

$$= x_0 + \int_0^t a(X_0)\,ds + \int_0^t b(X_0)\,dW_s + \int_0^t [a(X_s) - a(X_0)]\,ds + [b(X_s) - b(X_0)]\,dW_s$$

$t \downarrow 0$ $$= x_0 + \underbrace{a(X_0)\,t}_{O(t)} + b(X_0)\underbrace{(W_t - 0)}_{O(\sqrt{t})} + (\quad)$$

$a, b \in C_b^2$

$I_0 = \int_0^t s\,ds = \frac{1}{2}t^2$

$I_{10} = \int_0^t W_s\,ds = O(t^{3/2})$.

$\mathbb{E} I_{10}^2 = \mathbb{E} \int_0^t \int_0^t W_s W_r\,dr\,ds = \int_0^t \int_0^t s \wedge r\,dr\,ds = \frac{1}{6}t^3$

$$a(X_s) - a(X_0) = \int_0^s a'(X_r)\left[a(X_r)dr + b(X_r)dW_r\right] + \frac{1}{2}a'' b^2 dr$$

$$\int_0^t [a(X_s) - a(X_0)]\,ds = \int_0^t \int_0^s \Big\{ \underbrace{\left[a' a + \tfrac{1}{2} a'' b^2\right]}_{f} dr + \underbrace{a' b}_{g}\, dW_r \Big\}\,ds$$

$$= \left[a'a + \tfrac{1}{2}a''b^2\right](X_0)\,\tfrac{1}{2}t^2 + [a'b](X_0)\underbrace{\int_0^t (W_s - W_0)\,ds}_{O(t^{3/2})} + \int_0^t \int_0^s \left\{[f(X_r) - f(X_0)]\,dr + [g(X_r) - g(X_0)]\,dW_r\right\} ds \quad o(t^{3/2})$$

$$\int_0^t [b(X_s) - b(X_0)]\,ds = \int_0^t \int_0^s \Big\{ \underbrace{\left[b'a + \tfrac{1}{2} b'' b^2\right]}_{f} dr + \underbrace{b' b}_{g}\, dW_r \Big\}\,dW_s$$

$$= \left[b'a + \tfrac{1}{2}b''b^2\right](X_0)\underbrace{\int_0^t s\,dW_s}_{O(t^{3/2})} + [b'b](X_0)\underbrace{\int_0^t \int_0^s dW_r\,dW_s}_{O(t)} + \int_0^t \int_0^s \{\Delta f(X_r)\,dr + \Delta g(X_r)\,dW_r\}\,dW_s$$

$o(t) \to O(t^{3/2})$

EM: $$X_t = x_0 + a(x_0)t + b(x_0)(W_t - W_0) + o(\sqrt{t})$$

$$X_{t_{n+1}} = X_{t_n} + a(X_{t_n})\Delta t + b(X_{t_n})(W_{t_{n+1}} - W_{t_n}) + o(\sqrt{\Delta t})$$

$$\underline{Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n}$$

Milstein: $$X_{t_{n+1}} = \cdots\cdots\cdots + [b'b](X_{t_n}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_r\,dW_s + o(\Delta t).$$

$$\int_0^t \int_0^s dW_r\,dW_s = \int_0^t (W_s - W_0)\,dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t$$

$$\underline{Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n + [b'b](Y_n)\,\tfrac{1}{2}(\Delta W_n^2 - \Delta t)}.$$

IT1.5 $$X_t = x_0 + a(x_0)t + b(x_0)\underbrace{(W_t - W_0)}_{I_1} + [b'b](x_0)\underbrace{\int_0^t \int_0^s dW_r\,dW_s}_{I_{11}} + \left[b'a + \tfrac{1}{2}b''b^2\right](x_0)\underbrace{\int_0^t s\,dW_s}_{I_{01}}$$
$$+ [a'b](x_0)\underbrace{\int_0^t (W_s - W_0)\,ds}_{I_{10}} + \left[(b'b)'\,b\right](x_0)\underbrace{\int_0^t \int_0^s \int_0^r dW_u\,dW_r\,dW_s}_{I_{111}} + o(t^{3/2}) + \left[a'a + \tfrac{1}{2}a''b^2\right](x_0)\tfrac{1}{2}t^2$$

Attention: relation between multiple integrals: I_1, I_{11}, I_{10}, I_{111}, I_{01}

$I_{11} = \frac{1}{2}(I_1^2 - t)$ $\qquad$ $\mathbb{E}[I_{10}] = 0$; $\mathbb{E}[I_{10}^2] = \frac{1}{3}t^3$ $\qquad$ $\int_0^t \int_0^t \underbrace{\mathbb{E}[W_s W_r]}_{s \wedge r}\,dr\,ds$

$I_{111} = \frac{1}{3!}(I_1^3 - 3tI_1)$ $\qquad$ $\mathbb{E}[I_{10} I_1] = \frac{1}{2}t^2$ $\qquad$ $2\int_0^t \int_0^s r\,dr\,ds$

$I_{01} = tI_1 - I_{10}$ $\qquad$ $\int_0^t s\,dB_s = tB_t - \int_0^t B_s\,ds$ $\qquad$ $\mathbb{E}\left[\int_0^t W_s\,ds\, W_t\right] = \int_0^t s\,ds = \frac{1}{2}t^2$

$$\Delta Z = I_{10} = \int_0^t W_s\,ds$$

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n + [b'b](Y_n)\,\tfrac{1}{2}(\Delta W_n^2 - \Delta t)$$
$$+ [b'a + \tfrac{1}{2}b''b^2](Y_n)(\Delta t\,\Delta W_n - \Delta Z) + [a'b](Y_n)\Delta Z$$
$$+ [(b'b)'b](X_0)\,\tfrac{1}{6}(\Delta W_n^3 - 3\Delta t\,\Delta W_n) + [a'a + \tfrac{1}{2}a''b^2](Y_n)\,\tfrac{1}{2}\Delta t^2$$

$\Delta W_n = W_{t_{n+1}} - W_{t_n} = \sqrt{\Delta t}\,\xi_n$, $\Delta Z_n = \frac{1}{2}(\Delta t)^{3/2}(\xi_n + \frac{1}{\sqrt{3}}\eta_n)$, $\xi_n, \eta_n \sim N(0,1)$, i.i.d.

- When $b' = 0$, i.e. constant diffusion coefficient,

  EM = Milstein.

  IT1.5: $Y_{n+1} = Y_0 + a(Y_n)\Delta t + b(Y_n)\Delta W_n + [a'b](Y_n)\Delta Z_n + [a'a + \frac{1}{2}a''b](Y_n)\frac{1}{2}\Delta t^2$.

High-D: $x \in \mathbb{R}^n$.

EM ✓ $\quad \underline{Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n}$

Milstein: Derive from Ito's Chain rule, becoming complicated.

- In practice, hybrid schemes are often used, e.g. RK4 + EM for additive noise, Splitting methods for Hamiltonian eq.
  - The drift is nonlinear (local Lipschitz) and dominates the dynamics (stability ☆).
  - The order is limited by the lowest order part.

3. Strong/weak order of convergence

$$X_T \leftrightarrow Y_N^\delta \qquad \delta = \frac{T}{N}$$

strong order γ: $\mathbb{E}|X_T - Y_N^\delta| \le C\delta^\gamma$ for all $\delta \in (0, \delta_0)$.

weak order β: $|\mathbb{E}[g(X_T) - g(Y_N^\delta)]| \le C_g \delta^\beta$ - - - - - , $\forall g \in C^{2+2\beta}$.

$$dX_t = a(t,X_t)dt + b(t,X_t)dW_t$$

(<u>Thm 10.2.2</u>) EM has <u>strong order of convergence</u> $\gamma = 0.5$. if

(1) $\mathbb{E}[|X_0|^2] < \infty$; $\mathbb{E}[|X_0 - Y_0^\delta|^2] \le K_1\delta$

(2) Global Lipschitz & linear growth a.b.

(3) (Holder in time) $|a(s,x) - a(t,x)| + |b(s,x) - b(t,x)| \le K(1+|x|)|t-s|^{\frac{1}{2}}$.

(Thm 14.1.5) It has weak order $\beta = 1$ if g is smooth & $bb^T \geq \lambda I$ for some $\lambda > 0$.

(Thm 10.3.5) <u>Milstein has strong order</u> $\gamma = 1$. if (1) and (2-3) for a, b and derivatives.

weak order $\beta = 1$

About weak approximation, when only moments are of interest, can replace ΔW_n by $\Delta \hat{W}_n$

EM: $Y_{n+1} = Y_n + a\Delta t + b\Delta\hat{W}_n$, $\mathbb{P}(\Delta\hat{W}_n = \pm\sqrt{\Delta t}) = \frac{1}{2}$

$\begin{cases} \mathbb{E}\Delta\hat{W} = 0 = \mathbb{E}\Delta W \\ \mathbb{E}|\Delta\hat{W}|^2 = \Delta t = \mathbb{E}|\Delta W|^2 \end{cases}$ NOT high-order moments

Milstein: $Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta\hat{W}_n + [b'b](Y_n)(\frac{1}{2}\Delta\hat{W}_n^2 - \Delta t)$

<u>IT-2.0 weak</u>: $+ [b'a + \frac{1}{2}b''b^2](Y_n)(\Delta t\Delta\hat{W}_n - \Delta\hat{Z}) + [a'b](Y_n)\Delta\hat{Z}$

$\cancel{+ [(b'b)'b](Y_n)\frac{1}{6}(\Delta\hat{W}_n^3 - 3\Delta t\Delta\hat{W}_n)} + [a'a + \frac{1}{2}a''b](Y_n)\frac{1}{2}\Delta t^2$

$\Delta\hat{W}_n = W_{t_{n+1}} - W_{t_n} = \sqrt{\Delta}\,\xi_n$, $\Delta Z_n = \cancel{\frac{1}{2}(\Delta t)^{3/2}(\xi_n + \frac{1}{\sqrt{3}}\eta_n)}\ \frac{1}{2}(\Delta t)^{3/2}\xi_n$

<u>Order test</u>: How to do a numerical order test?

Note: $W_n^\delta = W_{t_{n+1}} - W_{t_n}$. $W_n^{k\delta} = W_{t_{(n+1)k\delta}} - W_{t_{nk\delta}} = \sum_{j=nk}^{(n+1)k}(W_{t_{j+1}} - W_{t_j})$

ΔZ_n? Need to consider: $\Delta Z_n = \int_{t_n}^{t_{n+1}} W_s\, ds$

4. Stability.

- convergence alone is no guarantee of "stability", particularly stiff DEs.
- Propagation of initial errors and roundoff errors.

4.1. Stability of the process / solution

Recall: ODE $\quad X'(t) = \lambda X(t)$; $\Rightarrow X(t) = X(0)e^{\lambda t}$; $\quad$ 0 is a steady state, it is $\boxed{\text{Stable}}$ if $\lambda < 0$ or $Re(\lambda)<0$.

$\lim_{t\to\infty} X(t) = 0, \ \forall X(0)$.

SDE ?

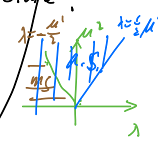
$$\begin{cases} \boxed{\text{Mean-square stable}}: & \lim_{t\to\infty} \mathbb{E}[X_t^2] = 0 \\ \boxed{\text{asymptotic stable}}: & \lim_{t\to\infty} X_t = 0 \quad a.s. \end{cases}$$

$\mu \downarrow 0$

$\Rightarrow \lambda < \frac{1}{2}\mu^2 \downarrow 0$

Example 1. $\quad dX_t = \lambda X_t dt + \mu X_t dW_t$; $\Rightarrow X_t = X_0 e^{(\lambda - \frac{1}{2}\mu^2)t + \mu W_t}$; $\quad$ 0 is a steady state.

$$\begin{cases} \mathbb{E} X_t^2 = \mathbb{E}[X_0^2 e^{2(\lambda-\frac{1}{2}\mu^2)t + 2\mu W_t}] = \mathbb{E}[X_0^2]\, e^{2(\lambda-\frac{1}{2}\mu^2)t + 2\mu^2 t} \downarrow 0 \text{ if } \lambda + \frac{1}{2}\mu^2 < 0. \\ X_t \to 0 \quad a.s. \qquad \text{if } \lambda - \frac{1}{2}\mu^2 < 0 \end{cases}$$



Stability of $\{X_n^{\Delta t}\}$ $\begin{cases} \boxed{\text{Mean-square stable}}: & \lim_{j\to\infty} \mathbb{E}[Y_j^2] = 0 \\ \boxed{\text{asymptotic stable}} & \lim_{j\to\infty} Y_j = 0 \ a.s. \end{cases}$

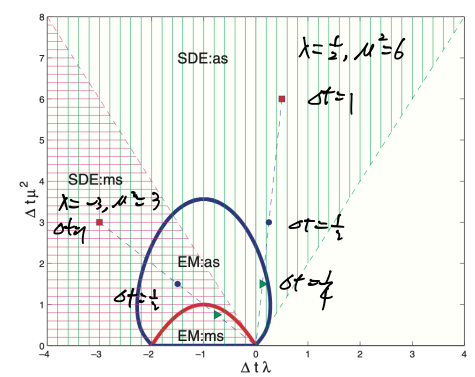


Fig. 6 *Mean-square and asymptotic stability regions.*

Stability regions

EM of Example 1. $\quad Y_{j+1} = (1+\lambda\Delta t)Y_j + \mu Y_j \Delta W_j$

① M.S.: $\mathbb{E}[Y_{j+1}^2] = (1+\lambda\Delta t)^2\, \mathbb{E}[Y_j^2] + \mu^2\Delta t\, \mathbb{E}[Y_j^2]$

$$= [(1+\lambda\Delta t)^2 + \mu^2\Delta t]^{j+1}\, \mathbb{E}[Y_0^2] \downarrow 0, \text{ if } |\cdot| < 1$$

② A.S.: $\quad Y_{j+1} = (1+\lambda\Delta t + \mu\Delta W_j)Y_j = \prod_{i=1}^{j+1}(1+\lambda\Delta t + \mu\Delta W_i)Y_0 \xrightarrow{j\to\infty} 0$

$\Leftrightarrow \prod_{i=1}^{j+1}|\cdot| \to 0 \Leftrightarrow \exp\left(\sum_{i=1}^{j+1} \log|1+\lambda\Delta t + \mu\Delta W_i|\right) \xrightarrow{j\to\infty} 0$

$\to Z_i \sim Z$

$$= \exp\left(\sum_{i=1}^{j+1}(Z_i - \mathbb{E}Z) + (j+1)\mathbb{E}Z\right)$$

$\doteq S_{j+1}$: Iterated law of Log: $\lim_{j\to\infty} \frac{|S_j|}{\sqrt{2j \log\log j}} = \sigma_Z$

$Z = \log|1+\lambda\Delta t + \mu\sqrt{\Delta t}N(0,1)|$

$\Leftrightarrow \mathbb{E}Z < 0.$

Numerical Stability regions: $\quad \Delta t$, parameter (λ, μ) or drift & diffusion

- The larger $|\lambda|$, the smaller Δt that we can tolerate.

Q: $dX_t = \lambda X_t dt + dW_t$, NO steady-state. How to define "stability"?

① Steady state → stationary solu. $\lim_{\delta\downarrow 0} \lim_{n\uparrow\infty} \mathbb{E}[|X_n^{\delta t}|^2] = \lim_{t\uparrow\infty} \mathbb{E}[X_t^2]$

② Numerical stability. KP92; §9.8

The appr. (Y_n^δ) of (Y) is $\boxed{\text{stochastically numerically stable}}$ if $\exists\, \Delta_0$ s.t., $\forall\, \delta\in(0,\Delta_0)$, $\forall \varepsilon>0$,

$$\lim_{|Y_0^\delta - Y_0|\to 0} \sup_{0\le n\delta\le T} \mathbb{P}(|Y_n^\delta - Y_{n\delta}| > \varepsilon) = 0. \qquad (1)$$

- - - - - - - $\boxed{\text{Asymptotically numerically stable}}$ if - - -

$$\lim_{|Y_0^\delta - Y_0|\to 0} \lim_{T\uparrow\infty} \mathbb{P}\Big(\sup_{0\le n\delta\le T} |Y_n^\delta - Y_{n\delta}| > \varepsilon\Big) = 0. \qquad (2)$$

Remark 1. Proof of (1) usually by controlling $\mathbb{E}[|Y_n^\delta - Y_{n\delta}|^2]$ & Markov inequality.

Proof of (2) usually by Doob's MG. inequality.

Remark 2: In practice, often interested in approximating the invariant measure; large δ → faster

Example: $dX_t = \lambda X_t dt + dW_t$ $\quad \lambda\in\mathbb{R},\ X_0\in\mathbb{R}$

EM $\quad Y_{n+1}^\delta = Y_n^\delta + \lambda Y_n^\delta\cdot\delta + W_n^\delta = (1+\lambda\delta) Y_n^\delta + W_n^\delta \qquad W_n^\delta \sim N(0,\delta)$

· if $\lambda>0$, then blowup, i.e. $|Y_{n+1}^\delta| \xrightarrow{n\uparrow\infty} \infty$, b.c. $Y_{n+1}^\delta = (1+\lambda\delta)^{n+1} Y_0^\delta + \underbrace{\sum_{j=1}^{n} (1+\lambda\delta)^j W_j^\delta}$ ("$\approx$" $O((1+\lambda\delta)^n)$)

· If $\lambda<0$ $\begin{cases} \text{if } |1+\lambda\delta|>1, & \text{blowup} \\ \text{if } |1+\lambda\delta|<1, & \text{then bounded.} \end{cases}$

$\Updownarrow$

$-1<\lambda\delta+1<1 \Leftrightarrow -2\le\lambda\delta\le 0 \Leftrightarrow \underline{\underline{0<\delta<-2/\lambda}}$

Implicit Euler: $Y_{n+1}^\delta = Y_n^\delta + \lambda Y_{n+1}^\delta \delta + W_n^\delta \Leftrightarrow Y_{n+1}^\delta = (1-\lambda\delta)^{-1}(Y_n^\delta + W_n^\delta)$

Thus, stable if $|1-\lambda\delta|>1$, → $\lambda<0$, $\underline{\delta>0 \text{ arbitrary!}}$

Invariant measure. $X_\infty \sim N(0, \frac{-1}{2\lambda})$ $\xleftarrow{\lambda<0}$ $X_t = e^{\lambda t}X_0 + \int_0^t e^{-\lambda(t-s)} dB_s$

EM: $Y_\infty^\delta \sim N(0, \frac{\delta}{1-(1+\lambda\delta)^2}) \to \frac{-1}{2\lambda - \lambda^2\delta}$

IE: $Y_\infty^\delta \sim N(0, \frac{(1-\lambda\delta)^{-2}\delta}{(1-(1-\lambda\delta)^{-2})}) \to \frac{\delta}{(1-\lambda\delta)^2 - 1} = \frac{-1}{2\lambda - \lambda^2\delta}$

b.c. $Z_{n+1} = \alpha Z_n + \beta W_n \Rightarrow Z_\infty \sim N(0,\sigma^2)$

$\sigma^2 = \mathbb{E}[Z_{n+1}^2] = \alpha^2 \mathbb{E}[Z_n^2] + \beta^2 \Rightarrow \sigma^2 = \frac{\beta^2}{1-\alpha^2}$

<u>Remark 2</u>: <u>Implicit Euler schemes</u>: $\quad Y_{n+1}^{\delta} = Y_n^{\delta} + a(t_{n+1}, Y_{n+1}^{\delta})\delta + b(t_n, Y_n^{\delta})\Delta W_n$

or $\quad$ - - - $\quad$ - - - $\quad b(t_{n+1}, Y_{n+1}^{\delta})\Delta W_n$

<u>Reading</u>:

- KP92: §6.3; §9.8, §12.5
- Mattingly + Higham + Stuart 02: Implicit schemes for local Lipschitz SDEs.