Numerical Approximation and Stability of SDEs

Time Discrete Approximations, Convergence, and Stability

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Time Discrete Approximations of SDEs

Consider the SDE:

$$dX_t = a(X_t)dt + b(X_t)dW_t, \quad X_{t_0} = x_0 \tag{1}$$

Discrete Times: $t_n = t_0 + n\Delta t$, for $n = 0, \ldots, N$, where $T = N\Delta t$.
Common Schemes:

- Euler-Maruyama (EM) Scheme:

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n, \quad Y_0 = X_0 \tag{2}$$

  where $\Delta W_n = W_{t_{n+1}} - W_{t_n} \sim N(0, \Delta t)$.

- Milstein Scheme:

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n + \frac{1}{2}bb'(Y_n)((\Delta W_n)^2 - \Delta t) \tag{3}$$

Key Questions in Numerical SDEs

When designing and analyzing numerical schemes for SDEs, we ask:

1. How to construct a "valid" scheme?
2. What is the order of convergence? (Strong vs. Weak)
3. How to construct higher-order schemes?
4. Schemes for high-dimensional SDEs?
5. Stability? (Long-term behavior)

Ito-Taylor Expansion (Wagner-Platen Formula)

Concept: Similar to Taylor expansion for ODEs, but uses Ito's formula: $df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2$.

Expansion of X_t with small t:

$$X_t = X_0 + a(X_0)\Delta t + b(X_0)\Delta W_t + \int_0^t [a(X_s) - a(X_0)]\, ds + \int_0^t [b(X_s) - b(X_0)]\, dW_s$$

Ito's Formula Application:

$$\begin{aligned} a(X_s) - a(X_0) &= \int_0^t \left[aa'(X_s) + \frac{1}{2}b^2a''(X_s)\right] ds + \int_0^t ba'(X_s)dW_s \\ b(X_s) - b(X_0) &= \int_0^t \left[ab'(X_s) + \frac{1}{2}b^2b''(X_s)\right] ds + \int_0^t bb'(X_s)dW_s \end{aligned} \tag{4}$$

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$I_1 = \int_0^t dW_s = W_t - W_0$, $\Delta W_n = W_{t_{n+1}} - W_{t_n} \sim N(0, \Delta t)$.

- Euler-Maruyama (EM) Scheme:

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n, \quad Y_0 = X_0$$

- Terms up to $a(X_0)\Delta t + b(X_0)\Delta W_t$, hence $O(\Delta t^{1/2})$.

Multiple Ito Integrals and higher-order schemes

Higher-order schemes require multiple Ito integrals.
$I_{01} = \int_0^t \int_0^s dW_r ds$.
$I_{01} = \int_t^{t+\Delta t} \int_t^s dW_r ds$, $I_{10} = \int_t^{t+\Delta t} \int_t^s ds dW_r$

- $I_1 = \int_0^t dW_s = W_t - W_0 \qquad \Leftrightarrow \Delta W_n \sim \mathcal{N}(0, \Delta t)$
- $I_{11} = \int_0^t \int_0^s dW_r dW_s = \frac{1}{2}(W_t^2 - t) \qquad \Leftrightarrow \frac{1}{2}((\Delta W_n)^2 - \Delta t)$
- $I_{01} = \int_0^t \int_0^s dr dW_s = tW_t - I_{10} \qquad \Leftrightarrow \Delta t \Delta W_n - Z_n$
- $I_{10} = \int_0^t \int_0^s dW_r ds \qquad \Leftrightarrow Z_n = \frac{1}{2}(\Delta t)[\Delta W_n + \frac{1}{\sqrt{3}}\eta_n]$, where $\eta_n \sim \mathcal{N}(0, \Delta t)$ independent of ΔW_n.

Milstein Scheme:

- Includes the $\frac{1}{2}bb'(Y_0)I_{11}$ term, order $O(\Delta t)$:

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n + \frac{1}{2}bb'(Y_n)((\Delta W_n)^2 - \Delta t)$$

- If $b(X_t) =$ constant, EM and Milstein schemes are identical.

Strong Order 1.5 Ito-Taylor Scheme

By including more terms from the Ito-Taylor expansion, we can achieve higher strong convergence orders.
$I_{111} = \int_0^t \int_0^s \int_0^r dW_u dW_r dW_s = \frac{1}{6}(I_1^3 - 3tI_1)$.
The strong order 1.5 scheme is:

$$
\begin{aligned}
Y_{n+1} = Y_n + a\Delta t + b\Delta W_n + \frac{1}{2}bb'((\Delta W_n)^2 - \Delta t) \\
+ (ab' + \frac{1}{2}b^2 b'')(\Delta W_n \Delta t - Z_n) + a'b(Y_n)Z_n \\
+ \frac{1}{2}(bb')'b(Y_n)\left(\frac{1}{3}(\Delta W_n)^2 - \Delta t\right)\Delta W_n \\
+ \frac{1}{2}(aa'(Y_n) + \frac{1}{2}b^2 a'')\Delta t^2
\end{aligned}
$$

Note: I_{10} and I_1 are correlated, so Z_n is used to account for this correlation. The strong order 1.5 scheme includes terms up to $O(\Delta t^{3/2})$.

High-Dimensional SDEs

For high-dimensional SDEs ($X \in \mathbb{R}^d$):

- Deriving Milstein schemes becomes significantly more complicated due to cross-terms in multiple Ito integrals (e.g., $\int W^{(i)} dW^{(j)}$).
- Hybrid schemes are often used. For example, using Runge-Kutta 4 (RK4) for the drift term and EM for additive noise:

$$
\begin{aligned}
k_1 &= a(Y_n) \\
k_2 &= a(Y_n + \frac{\Delta t}{2} k_1) \\
k_3 &= a(Y_n + \frac{\Delta t}{2} k_2) \\
k_4 &= a(Y_n + \Delta t k_3) \\
Y_{n+1} &= Y_n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) + b(Y_n)\Delta W_n
\end{aligned}
$$

- The overall order of a scheme is limited by its lowest-order part.

Strong vs. Weak Order of Convergence

Let X_T be the exact solution and Y_N be the numerical approximation at time $T = N\Delta t$.

- **Strong Order of Convergence (r):** Measures pathwise approximation.

$$\mathbb{E}[|X_T - Y_N|] \leq C\Delta t^r \tag{5}$$

  for some constant C and for all $\Delta t \in (0, \Delta t_0)$.

- **Weak Order of Convergence (p):** Measures approximation of the distribution (moments).

$$|\mathbb{E}[g(X_T)] - \mathbb{E}[g(Y_N)]| \leq C\Delta t^p \tag{6}$$

  for smooth test functions g.

Convergence Orders of Common Schemes

Euler-Maruyama (EM) Order:

- Strong order $r = 1/2$ if a, b satisfy global Lipschitz and linear growth conditions, and b is Hölder in time.
- Weak order $p = 1$ if g is smooth and bb' satisfies certain conditions.

Milstein Order:

- Strong order $r = 1$ if a, b and their derivatives satisfy certain regularity conditions.

Weak Approximation Note: When only moments (e.g., $\mathbb{E}[X_T]$ or $\mathbb{E}[X_T^2]$) are of interest, ΔW_n can be replaced by a simpler random variable (e.g., $\pm\sqrt{\Delta t}$ with probability 1/2).

Numerical Order Test

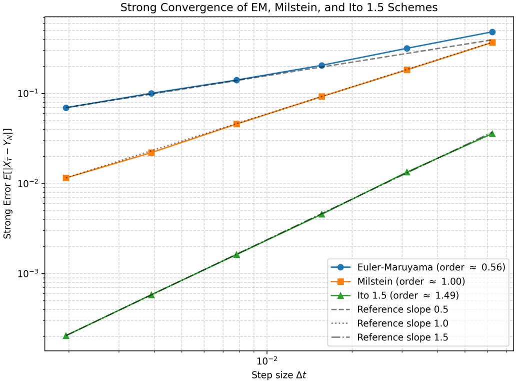
Estimating Convergence Order from Simulations:

- We can numerically estimate the order of convergence by running simulations with different step sizes Δt.
- For strong convergence, compute the error $e(\Delta t) = \mathbb{E}[|X_T - Y_N|]$ using a fine-grid reference solution if the exact solution is unknown.
- Plot $\log(e(\Delta t))$ vs. $\log(\Delta t)$. The slope of the line gives the empirical order of convergence r.
- Similarly, for weak convergence, plot $\log(|\mathbb{E}[g(X_T)] - \mathbb{E}[g(Y_N)]|)$ vs. $\log(\Delta t)$ to find the weak order p.

Numerical Test Results: Strong Convergence

Test Case: Geometric Brownian Motion $dX_t = 1.5X_t dt + 0.8X_t dW_t$.

- We simulate $M = 5000$ paths to estimate the strong error $\mathbb{E}[|X_T - Y_N|]$ at $T = 1$.

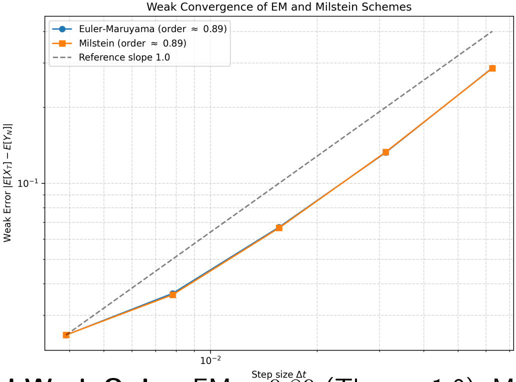


- **Empirical Strong Order:** EM ≈ 0.53 (Theory: 0.5), Milstein ≈ 0.96 (Theory: 1.0).

Numerical Test Results: Weak Convergence

Test Case: Geometric Brownian Motion, test function $g(x) = x$ (mean).

- We simulate $M = 500,000$ paths to estimate the weak error $|\mathbb{E}[X_T] - \mathbb{E}[Y_N]|$ at $T = 1$.



- **Empirical Weak Order:** EM ≈ 0.89 (Theory: 1.0), Milstein ≈ 0.89 (Theory: 1.0).
- **Note:** Milstein has the same weak order as EM because the expected value of its correction term is zero: $\mathbb{E}[(\Delta W_n)^2 - \Delta t] = 0$.