Homework assignment, week 5: numerical SDEs

1. Consider the SDE with smooth bounded a, b:

$$dX_t = a(X_t)dt + b(X_t)dW_t \tag{1}$$

Derive the strong order 1.5 Ito-Taylor scheme (see note or Kloeden+Platen Chapter 10). (Hint: you will need to use the multiple integrals and to pay attention to their relations:

$$I_1 = \int_0^h dW_s,\ I_{10} = \int_0^h \int_0^s dW_{s_1} ds,\ I_{11} = \int_0^h \int_0^s dW_{s_1} dW s,$$
$$I_{111} = \int_0^h \int_0^s \int_0^r dW_u dW_r dW s$$

Optional: write a code to test the order of convergence for your IT1.5 scheme.)

Solution to Problem 1. Let

$$L^0 f(x) = a(x)f'(x) + \frac{1}{2}b^2(x)f''(x), \qquad L^1 f(x) = b(x)f'(x).$$

Then, the Ito formula can be written as

$$f(X_t) - f(X_0) = \int_0^t L^0 f(X_s)ds + \int_0^t L^1 f(X_s)dW_s.$$

Applying to $a(X_s)$ and $b(X_s)$ in the integral form of the SDE gives

$$\begin{aligned}
X_h &= X_0 + \int_0^h a(X_s)ds + \int_0^h b(X_s)dW_s \\
&= X_0 + \int_0^h \left(a(X_0) + \int_0^s L^0 a(X_r)dr + \int_0^s L^1 a(X_r)dW_r \right) ds \\
&\quad + \int_0^h \left(b(X_0) + \int_0^s L^0 b(X_r)dr + \int_0^s L^1 b(X_r)dW_r \right) dW_s \\
&= X_0 + a_0 I_0 + b_0 I_1 + \int_0^h \int_0^s L^0 a(X_r)drds + \int_0^h \int_0^s L^1 a(X_r)dW_r ds \\
&\quad + \int_0^h \int_0^s L^0 b(X_r)drdW_s + \int_0^h \int_0^s L^1 b(X_r)dW_r dW_s,
\end{aligned} \tag{2}$$

where $a_0 = a(X_0)$, $b_0 = b(X_0)$, and we denote the multiple integrals by I_0, I_1, I_{00}, I_{10}, I_{01}, and I_{11} as follows:

$$I_0 = \int_0^h ds = h, \qquad I_{00} = \frac{h^2}{2}, \qquad I_1 = \int_0^h dW_s = W_h - W_0,$$
$$I_{10} = \int_0^h \int_0^s dW_{s_1} ds, \qquad I_{01} = \int_0^h \int_0^s ds_1 dW_s, \qquad I_{11} = \int_0^h \int_0^s dW_{s_1} dW s = \frac{1}{2}(I_1^2 - h),$$
$$I_{111} = \int_0^h \int_0^s \int_0^r dW_u dW_r dW s$$

The first two terms in (2) are of order h and $h^{1/2}$, respectively. The next four terms are of order h^2, $h^{1.5}$, $h^{1.5}$, and h, respectively. Hence, to get the strong order-1.5 scheme, we need to expand the integrand in the last term further, i.e., apply the Ito formula to $L^1 b(X_r)$ in the last term of (2):

$$\begin{aligned}\int_0^h \int_0^s L^1 b(X_r) dW_r dW_s &= \int_0^h \int_0^s \left((L^1 b)(X_0) + \int_0^r L^0 L^1 b(X_u) du + \int_0^r L^1 L^1 b(X_u) dW_u \right) dW_r dW_s \\ &= (L^1 b)_0 I_{11} + (L^1 L^1 b)_0 I_{111} + \text{higher order terms,}\end{aligned}$$

where the higher order terms include the order h^2 term $(L^0 L^1 b)_0 I_{011}$, with I_{011} defined as

$$I_{011} = \int_0^h \int_0^s \int_0^r ds dW_u dW_s.$$

Then, the Itô–Taylor expansion truncated at strong order 1.5 is

$$X_{t+h} = X_t + aI_0 + bI_1 + (L^0 a)I_{00} + (L^1 a)I_{10} + (L^0 b)I_{01} + (L^1 b)I_{11} + (L^1 L^1 b)I_{111} + R,$$

where all coefficients are evaluated at X_t (by replacing the integrals to be from t to $t+h$ instead of from 0 to h), and the remainder R contains only terms of higher strong order.

To get the numerical scheme, we replace the multiple integrals by their approximations. Using the relations between the multiple integrals

$$\begin{aligned} I_{10} &= hI_1 - I_{01}, \quad \text{because } I_{01} = \int_0^h s dW_s = hW_h - \int_0^h W_s ds, \\ I_{111} &= \frac{1}{6}(I_1^3 - 3hI_1), \end{aligned}$$

we obtain, with $\Delta W_n := I_1 = W_{t_{n+1}} - W_{t_n}$ and

$$J_n := I_{10} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_2} dW_{s_1} \, ds_2,$$

the order-1.5 scheme

$$\begin{aligned} Y_{n+1} = Y_n &+ a_n h + b_n \Delta W_n + \frac{1}{2}(L^0 a)_n h^2 + (L^1 a)_n J_n \\ &+ (L^0 b)_n (h\Delta W_n - J_n) + \frac{1}{2}(L^1 b)_n \big((\Delta W_n)^2 - h\big) \\ &+ \frac{1}{6}(L^1 L^1 b)_n \big((\Delta W_n)^3 - 3h\Delta W_n\big), \end{aligned}$$

where $a_n = a(Y_n)$, $b_n = b(Y_n)$, and similarly for the other terms.

Expanding the differential operators gives

$$\begin{aligned} L^0 a &= aa' + \frac{1}{2} b^2 a'', & L^1 a &= ba', \\ L^0 b &= ab' + \frac{1}{2} b^2 b'', & L^1 b &= bb', \\ L^1 L^1 b &= b\big((b')^2 + bb''\big). \end{aligned}$$

Hence, the scalar Itô–Taylor method of strong order 1.5 is

$$\begin{aligned} Y_{n+1} = Y_n + a_n h + b_n \Delta W_n + \frac{1}{2}\left(a_n a_n' + \frac{1}{2} b_n^2 a_n''\right) h^2 + b_n a_n' J_n \\ + \left(a_n b_n' + \frac{1}{2} b_n^2 b_n''\right)(h\Delta W_n - J_n) + \frac{1}{2} b_n b_n' \big((\Delta W_n)^2 - h\big) \\ + \frac{1}{6} b_n \big((b_n')^2 + b_n b_n''\big)\big((\Delta W_n)^3 - 3h\Delta W_n\big). \end{aligned}$$

In particular, to draw samples of J_n, we note that $J_n = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_2} dW_{s_1}\, ds_2$ is Gaussian with mean zero and variance $\mathbb{E}\left[J_n^2\right] = h^3/3$, but it is correlated with ΔW_n with correlation $\mathbb{E}\left[\Delta W_n J_n\right] = h^2/2$. Hence, we can sample J_n and ΔW_n by

$$\Delta W_n = \sqrt{h}\xi_n, J_n = \frac{h^{3/2}}{2}(\xi_n + \frac{1}{\sqrt{3}}\eta_n),$$

where ξ_n and η_n are independent standard normal random variables. ■

2. Consider the Ornstein-Uhlenbeck equation with $\lambda < 0$:

$$dX_t = \lambda X_t dt + \sigma dW_t,$$

(a) Find the range of the time step size δ such that the Euler-Maruyama scheme

$$Y_{n+1} = Y_n + \lambda Y_n \delta + \sigma\sqrt{\delta}\xi_n; \quad Y_0 = 0; \text{ where } \xi_n \sim \mathcal{N}(0,1)$$

is stable in the sense that $\mathbb{E}\left[Y_n^2\right] < \infty$ for all n and compute $\lim_{n\to\infty} \mathbb{E}\left[Y_n^2\right]$.
(b) Find the range of the time step size δ so that the implicit Euler scheme

$$Y_{n+1} = Y_n + \lambda Y_{n+1} \delta + \sigma\sqrt{\delta}\xi_n; \quad Y_0 = 0; \text{ where } \xi_n \sim \mathcal{N}(0,1)$$

is stable in the sense that $\mathbb{E}\left[Y_n^2\right] < \infty$ for all n and compute $\lim_{n\to\infty} \mathbb{E}\left[Y_n^2\right]$.

Solution to Problem 2. (a) The Euler–Maruyama scheme is

$$Y_{n+1} = (1 + \lambda\delta) Y_n + \sigma\sqrt{\delta}\,\xi_n.$$

Let $v_n := \mathbb{E}\left[Y_n^2\right]$. Since ξ_n is independent of Y_n, with $\mathbb{E}\left[\xi_n\right] = 0$ and $\mathbb{E}\left[\xi_n^2\right] = 1$,

$$v_{n+1} = (1 + \lambda\delta)^2 v_n + \sigma^2\delta.$$

Therefore,

$$v_n = \sigma^2\delta \sum_{k=0}^{n-1} (1 + \lambda\delta)^{2k}.$$

Thus, v_n remains bounded as $n \to \infty$ iff

$$|(1 + \lambda\delta)^2| < 1 \iff |1 + \lambda\delta| < 1.$$

Since $\lambda < 0$, this is equivalent to

$$\boxed{0 < \delta < -\frac{2}{\lambda}}.$$

In this case,

$$\lim_{n\to\infty} \mathbb{E}\left[Y_n^2\right] = \frac{\sigma^2\delta}{1-(1+\lambda\delta)^2} = \boxed{\frac{\sigma^2}{-2\lambda - \lambda^2\delta}}.$$

(b) The implicit Euler scheme can be written as

$$Y_{n+1} = \frac{Y_n + \sigma\sqrt{\delta}\,\xi_n}{1-\lambda\delta}.$$

Hence, the same calculation gives

$$v_{n+1} = \frac{v_n + \sigma^2\delta}{(1-\lambda\delta)^2}.$$

Since $\lambda < 0$, we have $1 - \lambda\delta > 1$ for every $\delta > 0$, so

$$0 < \frac{1}{(1-\lambda\delta)^2} < 1 \qquad \text{for all } \delta > 0.$$

Thus **the scheme is stable for every** $\delta > 0$. Taking limits in the recursion gives $v_\infty = \frac{v_\infty + \sigma^2\delta}{(1-\lambda\delta)^2}$, so

$$\lim_{n\to\infty} \mathbb{E}\left[Y_n^2\right] = \frac{\sigma^2\delta}{(1-\lambda\delta)^2 - 1} = \boxed{\frac{\sigma^2}{-2\lambda + \lambda^2\delta}}.$$

■