HW 4: Ito integral

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Exe 5(ii)–(iii). Verify that the given processes solve the given corresponding stochastic differential equations: (B_t denotes 1-dimensional Brownian motion)

(ii) $X_t = \frac{B_t}{1+t}$ with $B_0 = 0$ solves $dX_t = -\frac{X_t}{1+t}dt + \frac{1}{1+t}dB_t$ with $X_0 = 0$.

(iii) $X_t = \sin B_t$ with $B_0 = a \in (-\frac{\pi}{2}, \frac{\pi}{2})$ solves $dX_t = -\frac{1}{2}X_t dt + \sqrt{1-X_t^2}dB_t$ for $t < \tau := \inf\{s > 0 : B_s \notin [-\frac{\pi}{2}, \frac{\pi}{2}]\}$.

Solution. For (ii), note that $X_0 = 0$. By Itô product rule, we have

$$\begin{aligned} dX_t &= \frac{1}{1+t}dB_t + B_t d\left(\frac{1}{1+t}\right) + d\langle B_t, \frac{1}{1+t}\rangle \\ &= \frac{1}{1+t}dB_t - \frac{B_t}{(1+t)^2}dt \\ &= \frac{1}{1+t}dB_t - \frac{X_t}{1+t}dt. \end{aligned}$$

[One can also use Itô's formula to the function $f(x) = \frac{x}{1+t}$ to verify (ii).]

For (iii), when $t < \tau$, we have $B_t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, hence $\cos(B_t) > 0$ and $\sqrt{1-X_t^2} = \sqrt{1-\sin^2(B_t)} = \cos(B_t)$. Applying Itô's formula to $g(x) = \sin x$, we obtain for $t < \tau$,

$$\begin{aligned} dX_t = d(\sin B_t) &= g'(B_t)\, dB_t + \frac{1}{2}g''(B_t)\, dt \\ &= \cos(B_t)\, dB_t - \frac{1}{2}\sin(B_t)\, dt = \sqrt{1-X_t^2}\, dB_t - \frac{1}{2}X_t\, dt. \end{aligned}$$

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Exe 5.7. The *mean-reverting Ornstein-Uhlenbeck process* is the solution X_t of the SDE

$$dX_t = (m - X_t)dt + \sigma dB_t, \quad X_0 = x_0, \tag{1}$$

where $m \in \mathbb{R}$, $\sigma > 0$, and $B_t \in \mathbb{R}$.

(a) Solve this equation.

(b) Find $\mathbb{E}[X_t]$ and $\mathrm{Var}(X_t) = \mathbb{E}[(X_t - \mathbb{E}[X_t])^2]$.

Solution. (a) Using the integrating factor e^t, we have

$$\begin{aligned} d(e^t X_t) &= e^t dX_t + X_t de^t + d\langle e^t, X_t\rangle \\ &= e^t(m - X_t)dt + e^t\sigma dB_t + e^t X_t dt \\ &= me^t dt + e^t\sigma dB_t. \end{aligned}$$

Integrating both sides, we have

$$e^t X_t = X_0 + m\int_0^t e^s ds + \sigma\int_0^t e^s dB_s = X_0 + m(e^t - 1) + \sigma\int_0^t e^s dB_s.$$

Thus, we have

$$\boxed{X_t = e^{-t}X_0 + m(1-e^{-t}) + \sigma e^{-t}\int_0^t e^s dB_s.} \tag{2}$$

(b) By (2), we have

$$\mathbb{E}\left[X_t\right] = e^{-t}\mathbb{E}\left[X_0\right] + m(1-e^{-t}) + \sigma e^{-t}\mathbb{E}\left[\int_0^t e^s dB_s\right] = \boxed{e^{-t}\mathbb{E}\left[X_0\right] + m(1-e^{-t})}.$$

Using the fact that the stochastic integral has mean zero and is independent of the initial condition, we have that the variance is the sum of the variance of the two independent terms:

$$\mathrm{Var}(X_t) = e^{-2t}\mathrm{Var}(X_0) + \mathbb{E}\left[\left(\sigma e^{-t}\int_0^t e^s dB_s\right)^2\right] = \boxed{e^{-2t}\mathrm{Var}(X_0) + \frac{\sigma^2}{2}(1-e^{-2t})},$$

where we used the Itô isometry: $\sigma^2 e^{-2t}\mathbb{E}\left[\left(\int_0^t e^s dB_s\right)^2\right] = \sigma^2 e^{-2t}\int_0^t e^{2s}ds = \frac{\sigma^2}{2}(1-e^{-2t})$. ■

Exe 5.16(c). The technique used in Exercise 5.6 can be applied to more general nonlinear stochastic differential equations of the form

$$dX_t = f(t, X_t)\,dt + c(t)X_t\,dB_t, \qquad X_0 = x, \tag{5.3.11}$$

where $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R} \to \mathbb{R}$ are given continuous (deterministic) functions. Proceed as follows:

(a) Define the "integrating factor"

$$F_t = F_t(\omega) = \exp\left(-\int_0^t c(s)\,dB_s + \frac{1}{2}\int_0^t c^2(s)\,ds\right). \tag{5.3.12}$$

Show that (5.3.11) can be written

$$d(F_tX_t) = F_t \cdot f(t, X_t)\,dt. \tag{5.3.13}$$

(b) Now define

$$Y_t(\omega) = F_t(\omega)X_t(\omega), \tag{5.3.14}$$

so that

$$X_t = F_t^{-1}Y_t. \tag{5.3.15}$$

Deduce that equation (5.3.13) gets the form

$$\frac{dY_t(\omega)}{dt} = F_t(\omega)\,f\big(t, F_t^{-1}(\omega)Y_t(\omega)\big), \qquad Y_0 = x. \tag{5.3.16}$$

Note that this is just a deterministic differential equation in the function $t \mapsto Y_t(\omega)$, for each $\omega \in \Omega$. We can therefore solve (5.3.16) with ω as a parameter to find $Y_t(\omega)$ and then obtain $X_t(\omega)$ from (5.3.15).

(c) Apply this method to solve the stochastic differential equation

$$dX_t = \frac{1}{X_t}\,dt + \alpha X_t\,dB_t, \qquad X_0 = x > 0, \tag{5.3.17}$$

where α is constant.

(d) Apply the method to study the solutions of the stochastic differential equation

$$dX_t = X_t^{\gamma}\,dt + \alpha X_t\,dB_t, \qquad X_0 = x > 0, \tag{5.3.18}$$

where α and γ are constants. For what values of γ do we get explosion?

Solution for (a). By Itô's formula, we have

$$\begin{aligned} dF_t &= F_t\left(-c(t)dB_t + \frac{1}{2}c^2(t)dt\right) + \frac{1}{2}F_t c^2(t)dt \\ &= -F_t c(t)dB_t + F_t c^2(t)dt. \end{aligned}$$

Then, by Itô's product rule, we have

$$\begin{aligned} d(F_tX_t) &= F_t dX_t + X_t dF_t + d\langle F_t, X_t\rangle \\ &= F_t[f(t,X_t)dt + c(t)X_t dB_t] + X_t[-F_t c(t)dB_t + F_t c^2(t)dt] - F_t X_t c^2(t)dt \\ &= F_t f(t,X_t)dt. \end{aligned}$$

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Solution for (c). Here $f(t,x) = 1/x$ and $c(t) = \alpha$. Hence

$$F_t = \exp\Big(-\alpha B_t + \frac{1}{2}\alpha^2 t\Big).$$

By part (b), with $Y_t = F_t X_t$, we get the pathwise ODE

$$\frac{dY_t}{dt} = F_t\, f\big(t, F_t^{-1}Y_t\big) = \frac{F_t}{F_t^{-1}Y_t} = \frac{F_t^2}{Y_t}, \qquad Y_0 = x. \tag{3}$$

Therefore,

$$\frac{d}{dt}(Y_t^2) = 2Y_t\frac{dY_t}{dt} = 2F_t^2.$$

Integrating from 0 to t gives

$$Y_t^2 = x^2 + 2\int_0^t F_s^2\,ds = x^2 + 2\int_0^t \exp(-2\alpha B_s + \alpha^2 s)\,ds.$$

Since $Y_0 = x > 0$ and the right hand side of (3) is strictly positive, so Y_t is always positive:

$$Y_t = \left(x^2 + 2\int_0^t \exp(-2\alpha B_s + \alpha^2 s)\,ds\right)^{1/2}.$$

Using $X_t = F_t^{-1}Y_t$, we obtain

$$X_t = \exp\Big(\alpha B_t - \frac{1}{2}\alpha^2 t\Big)\left(x^2 + 2\int_0^t \exp(-2\alpha B_s + \alpha^2 s)\,ds\right)^{1/2}.$$

This solution stays strictly positive for all $t \geqslant 0$. ■