Doob's h-Transform: Exercise 7.14(b)

Problem statement. Let B_t be n-dimensional Brownian motion, let $D \subset \mathbb{R}^n$ be a bounded open set, and let $h > 0$ be harmonic on D, i.e.

$$\Delta h = 0 \qquad \text{in } D.$$

Consider the stochastic differential equation

$$dX_t = \nabla(\log h)(X_t)\, dt + dB_t. \tag{1}$$

Since the drift $\nabla(\log h)$ need not be globally well-behaved near ∂D, the solution is defined up to its lifetime by exhaustion of D. That is, choose an increasing sequence of open sets $\{D_k\}_{k \geq 1}$ such that $\overline{D_k} \subset D$ and $\bigcup_{k=1}^{\infty} D_k = D$. For each k, let

$$\tau_{D_k} := \inf\{t \geq 0 : X_t \notin D_k\}, \quad \tau := \lim_{k \to \infty} \tau_{D_k}.$$

Because $\nabla(\log h)$ is Lipschitz on $\overline{D_k}$, Eq.(1) has a unique strong solution on $[0, \tau_{D_k})$. By uniqueness these local solutions are compatible for different k, and therefore define a process X_t on $[0, \tau)$. This τ is the lifetime of the solution.

Assume now that there exists $x_0 \in \partial D$ such that

$$\lim_{x \to y,\ x \in D} h(x) = \begin{cases} 0, & y \in \partial D,\ y \neq x_0, \\ \infty, & y = x_0. \end{cases} \tag{2}$$

Prove that

$$\lim_{t \uparrow \tau} X_t = x_0 \qquad \text{a.s.}$$

Proof. We divide the proof into four steps.

Step 1: A harmonic test function for the generator. By part (a), the generator A of X_t satisfies

$$Au = \frac{\Delta(hu)}{2h}, \qquad u \in C_0^2(D). \tag{3}$$

In particular, if $u = 1/h$, then $Au = \frac{\Delta(hu)}{2h} = \frac{\Delta 1}{2h} = 0$.

Step 2: Excluding compact subsets of $\partial D \setminus \{x_0\}$. Let $F \subset \partial D \setminus \{x_0\}$ be compact. We show that

$$\mathbb{P}^x(E_F) = 0, \quad E_F := \{X_t \text{ has a cluster point in } F \text{ as } t \uparrow \tau\}. \tag{4}$$

Since $h(x) \to 0$ as $x \to y \in F$ from within D, we have

$$u(x) = \frac{1}{h(x)} \to \infty \qquad \text{as } x \to y \in F.$$

Then, for each $N \geq 1$, there exists an open neighborhood U_N of F in $\overline{D}$ such that

$$u(z) \geq N, \qquad z \in U_N \cap D.$$

Define the stopping time

$$\sigma_N := \inf\{t < \tau :\ X_t \in U_N \cap D\}.$$

For fixed k and $t > 0$, let

$$T_{t,N,D_k} := \sigma_N \wedge \tau_{D_k} \wedge t.$$

Since $Au = 0$, Dynkin's formula gives

$$\mathbb{E}^x[u(X_{T_{t,N,D_k}})] = u(x).$$

On the event $\{\sigma_N < \tau_{D_k},\ \sigma_N \le t\}$, we have $X_{T_{t,N,D_k}} = X_{\sigma_N} \in U_N \cap D$, so $u(X_{T_{t,N,D_k}}) \ge N$ and hence

$$u(x) = \mathbb{E}^x[u(X_{T_{t,N,D_k}})] \ge N\,\mathbb{P}^x(\sigma_N < \tau_{D_k},\ \sigma_N \le t).$$

Letting $t \to \infty$ yields

$$u(x) \ge N\,\mathbb{P}^x(\sigma_N < \tau_{D_k}).$$

Letting $k \to \infty$ and using the fact that $\tau_{D_k} \uparrow \tau$, we have

$$u(x) \ge N\,\mathbb{P}^x(\sigma_N < \tau).$$

If E_F in (4) occurs, then for every neighborhood U_N of F, the path must enter $U_N \cap D$ before time τ. Hence, $E_F \subset \{\sigma_N < \tau\}$ for every N. Using the previous inequality, we get

$$\mathbb{P}^x(E_F) \le \mathbb{P}^x(\sigma_N < \tau) \le \frac{u(x)}{N}, \qquad \forall N.$$

Letting $N \to \infty$, we obtain $\mathbb{P}^x(E_F) = 0$.

Step 3: The only possible boundary cluster point is x_0. For $m \ge 1$, define

$$F_m := \{y \in \partial D :\ |y - x_0| \ge 1/m\}.$$

Each F_m is compact and contained in $\partial D \setminus \{x_0\}$. By Step 2,

$$\mathbb{P}^x(X_t \text{ has a cluster point in } F_m \text{ as } t \uparrow \tau) = 0, \qquad m \ge 1.$$

Since $\partial D \setminus \{x_0\} = \bigcup_{m=1}^{\infty} F_m$, it follows that, almost surely, X_t has no cluster point in $\partial D \setminus \{x_0\}$.

Step 4: Conclusion. Because D is bounded, $\overline{D}$ is compact, so every path $\{X_t : 0 \le t < \tau\}$ has at least one cluster point in $\overline{D}$ as $t \uparrow \tau$. Moreover, no cluster point can lie in D: if $X_{t_j} \to z \in D$ along some sequence $t_j \uparrow \tau$, then for some ball $B(z,r) \Subset D$ the coefficients of (1) are smooth on $B(z,r)$, so the solution can be continued beyond τ, contradicting maximality of the lifetime. Hence every cluster point lies on ∂D.

Step 3 shows that the only possible boundary cluster point is x_0. Therefore, $\lim_{t\uparrow\tau} X_t = x_0$ a.s., and this completes the proof. ☐

Remark. *The diffusion with drift $\nabla \log h$ may be viewed as Brownian motion in D conditioned to exit the domain at the distinguished boundary point x_0.*

Example (A standard example of h)**.** *Let*

$$D = B(0,1) \subset \mathbb{R}^n, \qquad x_0 \in \partial B(0,1),\ |x_0| = 1,$$

and define

$$h(x) = \frac{1 - |x|^2}{|x - x_0|^n}, \qquad x \in B(0,1).$$

Up to a multiplicative constant, this is the Poisson kernel of the unit ball with pole at x_0. It satisfies:

- $h(x) > 0$ *for* $x \in B(0,1)$ *and* h *is harmonic in* $B(0,1)$*;*
- *if* $y \in \partial B(0,1)$ *and* $y \ne x_0$*, then* $h(x) \to 0$ *as* $x \to y,\ x \in B(0,1)$*;*

- *if* $x \to x_0$ *from within* $B(0,1)$*, then* $h(x) \to \infty$*.*

Thus h *satisfies condition* (2) *in the problem statement. In dimension* $n = 2$*, for instance with* $x_0 = (1,0)$*,* $h(x) = \frac{1-|x|^2}{|x-(1,0)|^2}$*.*

To verify directly that h *is harmonic in* $B(0,1)$*, write*

$$a := x_0, \qquad |a| = 1, \qquad f(x) := 1 - |x|^2, \qquad g(x) := |x-a|^{-n},$$

so that $h = fg$*. Since* $a \in \partial B(0,1)$*, the point* a *is not in the open ball, hence* g *is smooth on* $B(0,1)$*. Now, with* $r = |x-a|$*,*

$$\begin{aligned} &\nabla f(x) = -2x, \qquad \Delta f(x) = -2n, \\ &\nabla g(x) = -n\, r^{-n-2}(x-a), \qquad \Delta g(x) = 2n\, r^{-n-2}. \end{aligned}$$

Therefore, by the product rule,

$$\begin{aligned} \Delta h =& f\,\Delta g + 2\,\nabla f \cdot \nabla g + g\,\Delta f = 2n(1-|x|^2)r^{-n-2} + 4n\, x \cdot (x-a)\, r^{-n-2} - 2n\, r^{-n} \\ =& 2nr^{-n-2}\Big[(1-|x|^2) + 2x \cdot (x-a) - |x-a|^2\Big] = 0, \end{aligned}$$

because the term in the bracket is

$$1 - |x|^2 + 2|x|^2 - 2x \cdot a - \big(|x|^2 - 2x \cdot a + |a|^2\big) = 1 - |a|^2 = 0.$$