1. Kolmogorov's backward/forward equ.

Let X_t be an Ito diffusion w/ generator A. i.e. $dX_t = b(X_t)dt + \sigma(X_t)dB_t$; Lipschitz

$$Af = b\cdot\nabla f + \frac{1}{2}(\sigma\sigma)^T : \mathrm{Hess}(f), \quad \forall f \in C_0^2$$

$$= \lim_{t\to 0} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t}$$

Recall

Dynkin's formula: $\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x\left[\int_0^\tau Af(X_s)ds\right]$. for τ: stopping time, $\mathbb{E}^x[\tau] < \infty$

From Ito's formula:

$$\underbrace{\mathbb{E}^x[f(X_t)]}_{u(t,x)} = f(x) + \mathbb{E}^x\left[\int_0^t Af(X_s)ds\right] \qquad \partial_t u(0,x) = Au(0,x), \quad t=0$$

$$\Rightarrow \boxed{\frac{d}{dt}\mathbb{E}^x[f(X_t)] = \mathbb{E}^x[Af(X_t)].} \overset{??}{=} A\,\mathbb{E}^x[f(X_t)] \Rightarrow \partial_t u = Au, \quad t>0$$

<u>Theorem 8.1.1 (Kolmogorov's backward equ.)</u> Let $f \in C_0^2(\mathbb{R}^n)$, $u(t,x) = \mathbb{E}^x[f(X_t)]$

(a). Then $u(t,\cdot) \in \mathrm{Dom}(A)$ $\forall t$, $\begin{cases} \partial_t u = Au, & t>0, x\in\mathbb{R}^n \\ u(0,x) = f(x), & x\in\mathbb{R}^n. \end{cases}$

(b) If $w(t,x) \in C^{1,2}(\mathbb{R}\times\mathbb{R}^n)$ is bdd & satisfying $\begin{cases} \partial_t w = Aw \\ w(0,x) = f(x) \end{cases}$, then $w = \mathbb{E}^x[f(X_t)]$.

Proof: a)

$$Au(t,x) = \lim_{r\downarrow 0} \frac{\mathbb{E}^x[u(t,X_r)] - u(t,x)}{r} = \frac{1}{r}\mathbb{E}^x\left[\underbrace{\mathbb{E}^{X_r}[f(X_t)]}_{= \mathbb{E}^x[f(X_{t+r})|\mathcal{F}_r]} - f(X_t)\right] \quad \text{Markov.}$$

$$= \lim_{r\downarrow 0}\frac{1}{r}\mathbb{E}^x[f(X_{t+r}) - f(X_t)] \qquad \mathbb{E}^x[f(X_t)] = \mathbb{E}^x[\mathbb{E}^x[f(X_t)|\mathcal{F}_r]]$$

$$= \lim_{r\downarrow 0}\frac{1}{r}[u(t+r,x) - u(t,x)]$$

$$= \partial_t u. \qquad \#$$

Fix (s,x)

b) $\begin{cases} \tilde{A}w = -\partial_t w + Aw = 0 \\ w(0,x) = f(x) \end{cases}$ $\qquad \tilde{A}$: $Y_t = \begin{pmatrix} s-t \\ X_t^{0,x} \end{pmatrix}$ $dY_t = \begin{pmatrix} -dt \\ dX_t \end{pmatrix} = \begin{pmatrix} -1 \\ b(X_t) \end{pmatrix}dt + \begin{pmatrix} 0 \\ \sigma(X_t) \end{pmatrix}dB_t$

By Dynkin's formula,

$$\mathbb{E}^{s,x}[w(Y_{t\wedge\tau_R})] = w(s,x) + \mathbb{E}^{s,x}\left[\int_0^{t\wedge\tau_R}\tilde{A}w(Y_r)dr\right] = w(s,x)$$

$\hookrightarrow \tau_R = \inf\{t>0 : |X_t| \geq R\}$.

$R\uparrow\infty \downarrow$

$\mathbb{E}^{s,x}[w(Y_t)]$, $\qquad \forall t\geq 0$.

Choosing $t=s$:

$$w(s,x) = \mathbb{E}^{s,x}[w(Y_s)] = \mathbb{E}\left[w\left(\begin{pmatrix} 0 \\ X_s^{0,x} \end{pmatrix}\right)\right] = \mathbb{E}[f(X_s^{0,x})] = u(s,x). \quad \#$$

Rmk, semigroup: $u(t,\cdot) = Q_t f = \mathbb{E}^x[f(X_t)]$; $\partial_t u = \frac{d}{dt}Q_t f = Q_t(Af) = A(Q_t f)$

a linear operator $\Rightarrow Q_t = e^{At}$ (Need further explanation, b.c. A may be unbounded)

· If A bdd: $= \sum_{n=0}^\infty \frac{1}{n!}(At)^n$

Exe 8.3 (Kolmogorov's forward Equ.) (Fokker-Planck Equ.)

Assume that X_t^x has a density $p_t(x,\cdot)$: $\mathbb{E}^x[f(X_t)] = \int f(y) p_t(x,y)\, dy$, $f \in C_0^2$

If $p_t(x,\cdot)$ is smooth, then $\partial_t p_t(x,y) = A_y^* p_t(x,y)$, $\forall x, y$.

$$A_y^* \phi = -\sum_i \partial_{y_i}(b_i \phi) + \sum_{ij} \partial_{y_i y_j}(a_{ij}\phi), \quad \forall \phi \in C^2.$$

Proof: By Dynkin's formula: $\mathbb{E}^x[f(X_t)] = f(x) + \mathbb{E}^x[\int_0^t Af(X_s)\, ds]$

$$\int_{\mathbb{R}^n} f(y) p_t(x,y)\, dy = f(x) + \int_{\mathbb{R}^n} \int_0^t Af(y) p_s(x,y)\, ds\, dy.$$

$\frac{d}{dt}$: $\langle f, \partial_t p_t \rangle_{L^2(\mathbb{R}^n)} = \langle Af, p_t \rangle = \langle f, A^* p_t \rangle$. #.

How to solve the PDE? **Does it have an equilibrium and what is it? Au = f?** How is it related to invariant measure?

The Resolvent $(\alpha I - A)^{-1}$

Def: (Resolvent Operator) $R_\alpha g(x) = \mathbb{E}^x[\int_0^\infty e^{-\alpha t} g(X_t)\, dt]$, $\forall g \in C_b(\mathbb{R}^n)$

(formally: $\int_0^\infty e^{-\alpha t} e^{At} g\, dt = \int_0^\infty e^{-(\alpha I - A)t} dt\, g = (\alpha I - A)^{-1} g$)

Lemma (Feller cts) Let $g \in C_b(\mathbb{R}^n)$. Then $u(t,x) = \mathbb{E}^x[g(X_t)]$ is cts.

$\Rightarrow$ $R_\alpha g$ is bdd & cts.

Theorem 8.1.5 (a) If $f \in C_0^2(\mathbb{R}^n)$, then $R_\alpha(\alpha - A) f = f$, $\forall \alpha > 0$.

(b) If $g \in C_b(\mathbb{R}^n)$, then $R_\alpha g \in \mathrm{Dom}(A)$ and $(\alpha - A) R_\alpha g = g$, $\forall \alpha > 0$.

Proof, (a) by Dynkin's formula, $R_\alpha(\alpha - A) f = \alpha R_\alpha f - R_\alpha A f = \alpha \int_0^\infty e^{-\alpha t} \mathbb{E}^x[f(X_t)]\, dt - \int_0^\infty e^{-\alpha t} \mathbb{E}^x[Af(X_t)]\, dt$

$$= -e^{-\alpha t} u(t,x)\Big|_0^\infty + \underbrace{\int_0^\infty e^{-\alpha t} \partial_t u(t,x)\, dt - \int_0^\infty e^{-\alpha t} Au\, dt}_{=0}$$

$$= f(x).$$

b). Need $\partial_t \underline{\mathbb{E}^x[R_\alpha g(X_t)]} = A[\,]$:

$$\mathbb{E}^x[R_\alpha g(X_t)] = \mathbb{E}^x[\mathbb{E}^{X_t}[\int_0^\infty e^{-\alpha s} g(X_s)\, ds]] \overset{\text{Markov}}{=} \mathbb{E}^x[\mathbb{E}^x[\int_0^\infty e^{-\alpha s} g(X_{t+s})\, ds \mid \mathcal{F}_t]]$$

$$= \mathbb{E}^x[\int_0^\infty e^{-\alpha s} g(X_{t+s})\, ds] = \int_0^\infty e^{-\alpha s} \underline{\mathbb{E}^x[g(X_{t+s})]}\, ds = \frac{d}{ds}\int_t^{t+s} \mathbb{E}^x[g(X_v)]\, dv$$

$$\frac{d}{dt} \qquad \overset{IBP}{=} \alpha \int_0^\infty e^{-\alpha s} \int_t^{t+s} \mathbb{E}^x[g(X_v)]\, dv\, ds$$

$$A R_\alpha g = \frac{d}{dt} \mathbb{E}^x[R_\alpha g(X_t)]\Big|_{t=0} = \alpha \int_0^\infty e^{-\alpha s}\, \mathbb{E}^x[g(X_{t+s})] - \mathbb{E}^x[g(X_t)]\Big|_{t=0} ds = \alpha R_\alpha g - g. \quad \#.$$