PDEs and Applications, take-home Midterm Exam

Your name: ____________________

This is an open-book take-home Exam, and you are supposed to complete the exam without getting help from others. Please show your work or explain how you reach your answers.

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1. (32 points). Recall that the Fourier sine series of π is

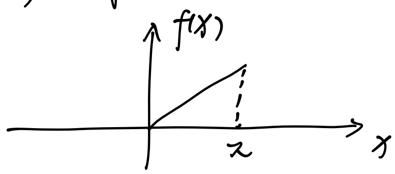
$$\pi \sim \sum_{n\geq 1, n \text{ odd}} \frac{4}{n}\sin(nx), \text{ for } x \in [0,\pi].$$

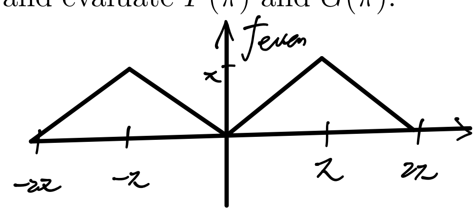
Let $f(x) = x$ for $x \in [0,\pi]$ and denote its Fourier cosine series and sine series by

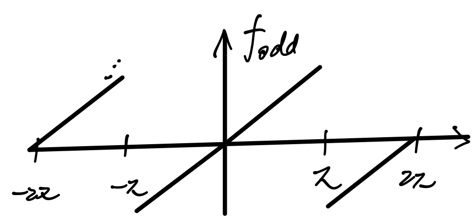
$$F(x) = \sum_{n=0}^{\infty} A_n \cos(nx); \quad G(x) = \sum_{n=1}^{\infty} B_n \sin(nx).$$

10 (a) Sketch the even & odd periodic extensions of f on $[-2\pi, 2\pi]$ and evaluate $F(\pi)$ and $G(\pi)$.
10 (b) Determine the coefficients A_n for all $n = 0, 1, 2, \ldots$.
12 (c) Evaluate the infinite series $\frac{4}{\pi}(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots)$.

Solution. (a) $f(x) = x$, $x \in [0,\pi]$ even periodic Extension







$F(\pi) = \frac{1}{2}[f_{even}(\pi-) + f_{even}(\pi+)] = \boxed{\pi}$

$G(\pi) = \frac{1}{2}[f_{odd}(\pi-) + f_{odd}(\pi+)] = \boxed{0}$

(b). $A_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \boxed{\frac{\pi}{2}}$

$A_n = \frac{2}{\pi}\int_0^{\pi} x \cdot \underbrace{\cos nx\,dx}_{=\frac{1}{n}d\sin nx} = \frac{2}{\pi n}\Big[\underbrace{x\sin nx\Big|_0^{\pi}}_{0} - \underbrace{\int \sin nx\,dx}_{+\frac{1}{n}\cos nx|_0^{\pi}}\Big]$

$= \boxed{\frac{2}{\pi}\frac{1}{n^2}[(-1)^n - 1]} = \begin{cases} 0, & n \text{ even} \\ -\frac{4}{\pi}\frac{1}{n^2}, & n \text{ odd} \end{cases}$

(c). f_{even} is continuous, so $f_{even}(x) = F(x)$.

$F(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi}\underbrace{\frac{1}{n^2}[(-1)^n - 1]}\cos nx$

$0 = f_{even}(0) = F(0) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\geq 1,\, n \text{ odd}}^{\infty} \frac{1}{n^2} \Rightarrow \frac{4}{\pi}(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots) = \boxed{\frac{\pi}{2}}$

2. (34 points) Solve the initial boundary value problem:

$$\begin{cases} \partial_t u = \partial_{xx} u + \frac{x}{\pi} + e^t \sin x, & \text{for } 0 < x < \pi, t > 0; \\ u(0,t) = 1, u(\pi,t) = t; \\ u(x,0) = 1 - \frac{x}{\pi}, & \text{for } 0 \le x \le \pi. \end{cases}$$

Ans: ① BC Homogenization: $u = v + w$

$$\begin{cases} \partial_{xx} w = 0 \\ w(0) = 1, \ w(\pi) = t \end{cases} \Rightarrow w(x,t) = 1 + \frac{t-1}{\pi} x.$$

Since $v = u - w$, we have

$$\partial_t v = \partial_t (u - w) = \partial_t u - \frac{x}{\pi} = \partial_{xx} u + \sin x e^t + \cancel{\frac{x}{\pi}} - \cancel{\frac{x}{\pi}}$$

$$\partial_{xx} v = \partial_{xx}(u - w) = \partial_{xx} u \qquad = \partial_{xx} v + \sin x e^t$$

$v(0,t) = u(0,t) - w(0,t) = 0, \quad v(\pi,t) = 0.$

$v(x,0) = u(x,0) - w(x,0) = 1 - \frac{x}{\pi} - (1 + \frac{-1}{\pi} x) = 0.$

Then $v(x,t)$ is the solution to the problem

$$(*) \begin{cases} \partial_t v = \partial_{xx} v + \sin x e^t, & x \in (0,\pi), t > 0 \\ v(0,t) = v(\pi,t) = 0 \\ v(x,0) = 0, & x \in (0,\pi) \end{cases}$$

② Solve $v(x,t)$ by eigenfunction expansion method.

$$\begin{cases} \varphi'' = -\lambda \varphi & x \in [0,\pi] \\ \varphi(0) = \varphi(\pi) = 0 \end{cases} \longrightarrow \begin{cases} \lambda_n = n^2 \\ \varphi_n(x) = \sin nx \end{cases}$$

Seek solution $v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin nx$

From the equ $(*)$ we get

$$\sum_1^{\infty} v_n'(t) \sin nx = \sum_1^{\infty} v_n(t)(-n^2) \sin nx + \sin x e^t$$

By orthogonality: $v_n'(t) = -n^2 v_n \quad n > 1$

$v_1'(t) = -v_1 + e^t$

$v_n(t) = e^{-n^2 t} v_n(0), \quad n > 1$

$v_1(t) = e^{-t} v_1(0) + \int_0^t e^{-(t-s)} e^s \, ds$

$= e^{-t} v_1(0) + \frac{1}{2}(e^t - e^{-t})$

Since $0 = v(x,0)$, we have $v_n(0) = 0$, $n = 0, \ldots$

Therefore, $v_n(t) \equiv 0$, $v_1(t) = \frac{1}{2}(e^t - e^{-t})$

Then, $v(x,t) = v_1(t) \sin x = \frac{1}{2}(e^t - e^{-t}) \sin x$

③ The solution is

$$\boxed{\begin{aligned} u(x,t) &= v(x,t) + w(x,t) \\ &= \frac{1}{2}(e^{-t} - e^{-t}) \sin x + 1 + \frac{t-1}{\pi} x. \end{aligned}}$$

3. (34 points) Find a solution, if exists, to the initial boundary value problem:

$$\begin{cases} \partial_{tt}u = 4\partial_{xx}u, & \text{for } 0 < x < \pi, t > 0; \\ \partial_x u(0,t) = 0, \ \partial_x u(\pi,t) = 0; \\ u(x,0) = \cos x, \ \partial_t u(x,0) = 0. \end{cases}$$

If a solution does not exist, explain why.

<u>Solution</u>: either separation of variables or <u>eigen-fn. expansion</u>.

10 1. Eigen function: $\begin{cases} \varphi'' = -\lambda\varphi \\ \varphi'(0)=0, \ \varphi'(\pi)=0 \end{cases} \Rightarrow \begin{array}{l} \lambda_n = n^2 \\ \varphi_n(x) = \cos nx. \end{array}$

2. Seek solution $u(x,t) = \sum_{n=0}^{\infty} a_n(t)\cos nx$

Assuming conditions for TBTD, $\partial_{tt}u = \sum_0^\infty a_n''(t)\cos nx$

$\partial_{xx}u = \sum_{n=1}^{\infty} a_n(t)(-n^2)\cos nx$

12 $\partial_{tt}u = 4\partial_{xx}u \Rightarrow a_n''(t) = -4n^2 a_n \Rightarrow a_n(t) = c_1\cos(2nt) + c_2\sin(2nt)$

$\partial_t u(x,0) = 0 \Rightarrow a_n'(0) = 0 \Rightarrow c_2 = 0$

12 $u(x,0) = \cos x \Rightarrow$ $a_0(0) = 0 \Rightarrow c_1 = 0, \ a_0(t) \equiv 0$

$a_1(0) = 1 \Rightarrow c_1 = 1, \ a_1(t) = \cos(2t)$

$a_n(0) = 0, \ n \geq 2 \Rightarrow c_1 = 0, \ a_n(t) = 0$

Thus, $u(x,t) = \cos x \cos(2t)$.