Chapter 3: Fourier series

Fei Lu

Department of Mathematics, Johns Hopkins

Outline

Section 3.4 Term-by-term differentiation

f on $[0, L]$ or $[-L, L]$:

- sine series: TBTD if f, f' are PS, f continuous and $f(L) = f(0) = 0$.
- cosine series: TBTD if f, f' are PS, f continuous.
- Fourier series: TBTD if f, f' are PS, f continuous and $f(L) = f(-L)$

Q: how did we prove it? Integration by parts + definition $\widetilde{f'}$:

$$f'(x) \sim \frac{1}{L}[f(L) - f(0)] + \sum_{n=1}^{\infty} \left[\frac{n\pi}{L} b_n + \frac{2}{L}[(-1)^n f(L) - f(0)] \right] \cos \frac{n\pi x}{L}$$

Method of eigenfunction expansion (generalizing separation of variables) Seek solution of the form

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos \frac{n\pi}{L} x + b_n(t) \sin \frac{n\pi}{L} x,$$

- PDE+ BC determines the eigenfunctions to use
- works for equation with source $\partial_t u = \kappa \partial_{xx} u + Q(x,t)$
- solve $a_n(t), b_n(t)$ from the PDE + IC (Assuming $\int \sum = \sum \int$)

Q: what about BC? Why solution in the form of Fourier series?

Outline

Section 3.5 Term-by-term Integration

Theorem

Let f be a piece-wise smooth function. We can always do TBTI of f's Fourier series and the resulted series always converge to the integral of f on $[-L, L]$. That is, ($\int \sum = \sum \int$)

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

$$\int_{-L}^{x} f(y)dy = a_0(x+L) + \sum_{n=1}^{\infty} \int_{-L}^{x} (a_n \cos \frac{n\pi y}{L} + b_n \sin \frac{n\pi y}{L})dy$$

Section 3.5 Term-by-term Integration

Theorem

Let f be a piece-wise smooth function. We can always do TBTI of f's Fourier series and the resulted series always converge to the integral of f on $[-L, L]$. That is, ($\int \sum = \sum \int$)

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

$$\int_{-L}^{x} f(y)dy = a_0(x+L) + \sum_{n=1}^{\infty} \int_{-L}^{x} (a_n \cos \frac{n\pi y}{L} + b_n \sin \frac{n\pi y}{L})dy$$

- Even if the original Fourier series has jump discontinuities
- The new series is continuous. Is it a Fourier series?

$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$

$$\int_{-L}^{x} f(y)dy - a_0(x+L) \text{``}=\text{''} \sum_{n=1}^{\infty} \int_{-L}^{x} (a_n \cos \frac{n\pi y}{L} + b_n \sin \frac{n\pi y}{L})dy \tag{1}$$

Proof(Basic idea: match the series. Shift $a_0(x+L)$ to get Fourier series)

Let $G(x) =$ LHS $= F(x) - a_0(x+L)$. We verify (1).

1. Note that $G(x)$ is PS+continuous, $G'(x) = f(x) - a_0$, $G(-L) = 0 = G(L)$.

$\Rightarrow G(x)$= its Fourier series= $A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L}$

2. Verify (1) by showing the coefficients are the same.

$$RHS = \sum_{n=1}^{\infty} a_n \frac{L}{n\pi} \sin \frac{n\pi x}{L} - b_n \frac{L}{n\pi} [\cos \frac{n\pi x}{L} - \cos(n\pi)]$$

$$A_n = \frac{1}{L} \int_{-L}^{L} G(x) \cos \frac{n\pi x}{L} dx = \cdots \text{integration by parts} = -b_n \frac{L}{n\pi}$$

$$B_n = \frac{1}{L} \int_{-L}^{L} G(x) \sin \frac{n\pi x}{L} dx = \cdots = a_n \frac{L}{n\pi}$$

$$A_0 = G(L) - \text{other terms} = \cdots = -\sum_{n=1}^{\infty} A_n \cos(n\pi)$$

** [compute A_0 from $G(L) = 0$, not $\frac{1}{2L} \int_{-L}^{L} G(x)dx$]

Fourier series: Convergence + TBTD + TBTI $\Rightarrow$ A new world

Example 1: evaluate $1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$ by Fourier sine series of $f(x)=1$: $\quad 1 \sim \frac{4}{\pi}\sum_{n=1,\text{ odd}}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{L}, \quad x\in[0,L]$.

Example 2: show that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.
(Hint: use $x\sim 2\sum_{n=1}^{\infty}\frac{L}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{L}$ and L^2)

Fourier series: Convergence + TBTD + TBTI $\Rightarrow$ A new world

Example 1: evaluate $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ by Fourier sine series of $f(x) = 1$: $1 \sim \frac{4}{\pi} \sum_{n=1,\text{ odd}}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{L}, \quad x \in [0, L]$.

Example 2: show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.
(Hint: use $x \sim 2\sum_{n=1}^{\infty} \frac{L}{n\pi}(-1)^{n+1} \sin \frac{n\pi x}{L}$ and L^2)

$$
\begin{aligned}
\frac{L^3}{3} = \int_0^L x^2 dx &= \int_0^L 4 \sum_{n,m=1}^{\infty} \frac{L^2}{mn\pi^2}(-1)^{n+m} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx \\
&= \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^L \sin \frac{n\pi x}{L} \sin \frac{n\pi x}{L} dx \text{ (Exchange of order)} \\
&= \frac{2L^3}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}
\end{aligned}
$$

Outline

Section 3.6 Complex form of Fourier series

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta; \quad \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}),\ \sin\theta = \frac{1}{2}(e^{i\theta} - e^{-i\theta}),$$

$$\begin{aligned} f(x) &\sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \\ &= \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{n\pi x}{L}}, \end{aligned}$$

where $c_0 = a_0, c_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2L}\int_{-L}^{L} f(x)[\cos\frac{n\pi x}{L} + i\sin\frac{n\pi x}{L})dx$. In short, $c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{i\frac{n\pi x}{L}} dx$ for all n.

Do we have orthogonality for $\{\phi_n(x) = e^{i\frac{n\pi x}{L}}\}$? (i.e. $\langle \phi_n, \phi_m \rangle = \delta_{n-m}$) $\downarrow$

Complex Orthogonality $\{\phi_n(x) = e^{i\frac{n\pi x}{L}}\}$

$$\langle \phi_n, \phi_m \rangle = \frac{1}{2L}\int_{-L}^{L} \phi_n(x)\overline{\phi_m(x)}dx = \frac{1}{2L}\int_{-L}^{L} e^{i\frac{(n-m)\pi x}{L}}dx = \delta_{n-m}$$

(note the complex conjugate)

Complex form of Fourier series

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{n\pi x}{L}}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{i\frac{n\pi x}{L}}dx$$

If f is a real-valued function, $c_{-n} = \overline{c_n}$

Summary of Chp 3: Fourier series

- Fourier series, sine/cosine/complex FS
- Fourier Theorem: convergence of FS

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x) = \frac{1}{2} \left[f(x^-) + f(x^+) \right]$$

- TBTD:
  - sine series: if f, f' are PS, f continuous and $f(L) = f(0) = 0$.
    $f'(x) \sim \frac{1}{L}[f(L) - f(0)] + \sum_{n=1}^{\infty} \left[\frac{n\pi}{L} b_n + \frac{2}{L}[(-1)^n f(L) - f(0)] \right] \cos \frac{n\pi x}{L}$
  - cosine series: if f, f' are PS, f continuous.
  - FS: if f, f' are PS, f continuous and $f(L) = f(-L)$
- TBTI: always! (if f PS)

- Enable us to treat infinite series!
  - compute series
  - method of eigenfunctions (non-homogeneous PDEs)

3.4.6. There are some things wrong in the following demonstration. Find the mistakes and correct them.

In this exercise we attempt to obtain the Fourier cosine coefficients of e^x:

$$e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}. \tag{3.4.22}$$

Differentiating yields

$$e^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin \frac{n\pi x}{L},$$

the Fourier sine series of e^x. Differentiating again yields

$$e^x = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 A_n \cos \frac{n\pi x}{L}. \tag{3.4.23}$$

Since equations (3.4.22) and (3.4.23) give the Fourier cosine series of e^x, they must be identical. Thus,

$$\left.\begin{aligned} A_0 &= 0 \\ A_n &= 0 \end{aligned}\right\} \text{ (obviously wrong!).}$$

By correcting the mistakes, you should be able to obtain A_0 and A_n *without* using the typical technique, that is, $A_n = 2/L \int_0^L e^x \cos n\pi x/L \, dx$.

Solution to 3-4-6:

Solution 3.4.6 $\quad e^x = A_0 + \sum_1^\infty A_n \cos\frac{n\pi}{L}x \qquad (1)$

Mistake: TBTD $\to$ $e^x \sim -\sum_1^\infty A_n \frac{n\pi}{L}\sin\frac{n\pi}{L}x$ NOT equality

Also, $e^L \neq 0$, $e^0 \neq 0$ $\Rightarrow$ can NOT TBTD again.

Instead, recall that $\left[\text{if } f, f' \text{ piecewise smooth, } f(x) \sim \sum_1^\infty B_n \sin\frac{n\pi}{L}x, \text{ then } f'(x) \sim \frac{1}{L}[f(L)-f(0)] + \sum_1^\infty \left\{\frac{n\pi}{L}B_n + \frac{2}{L}[(-1)^n f(L) - f(0)]\right\}\cos\frac{n\pi}{L}x\right]$

we have $e^x = (e^x)' \sim \frac{1}{L}[e^L - e^0] + \sum_1^\infty \left\{\frac{n\pi}{L}\left(-A_n\frac{n\pi}{L}\right) + \frac{2}{L}[(-1)^n e^L - e^0]\right\}\cos\frac{n\pi}{L}x \qquad (2)$

$\xrightarrow{(1)\,\&\,(2)}$ $\boxed{A_0 = \frac{1}{L}[e^L - e^0]}$

$A_n = -A_n\left(\frac{n\pi}{L}\right)^2 + \frac{2}{L}[(-1)^n e^L - e^0] \Rightarrow \boxed{A_n = \frac{2}{L}[(-1)^n e^L - e^0] \Big/ \left(1+\left(\frac{n\pi}{L}\right)^2\right).}$