Chapter 2: Method of Separation of Variables

Fei Lu

Department of Mathematics, Johns Hopkins

Solution to the IBVP?

$$\partial_t u = \kappa \partial_{xx} u + Q(x,t), \quad \text{with } x \in (0,L), t \geq 0$$
$$u(x,0) = f(x)$$
$$\text{BC: } u(0,t) = \phi(t), u(L,t) = \psi(t)$$

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Recall ODEs:

$$\underbrace{ay'' + by' + cy}_{\mathbf{L}y} = g(x); \quad y(x_0) = \alpha; y(x_1) = \beta.$$

- Step 1: solve the **linear** equation $\mathbf{L}y = 0 \Rightarrow y_1(x), y_2(x)$
- Step 2: find the specific solution $\mathbf{L}y = g \Rightarrow y_s(x)$

$\Rightarrow$ general solution: $y = c_1 y_1 + c_2 y_2 + y_s$ with c_1, c_2 TBD by BC/IC.

Same for PDE? key principles?

linear homogeneous $\Rightarrow$ Principle of Superposition (PoS)

Outline

Section 2.2: Linearity

Linear operator: for any $c_1, c_2 \in \mathbb{R}$,

$$\mathbf{L}(c_1 u_1 + c_2 u_2) = c_1 \mathbf{L}(u_1) + c_2 \mathbf{L}(u_2), \quad \forall u_1, u_2 \in Dom(\mathbf{L})$$

Examples: which operator(s) is nonlinear?

A. $\mathbf{L} = \partial_{xxx}$;

B. $\mathbf{L} = \partial_t - \kappa \partial_{xx}$;

C. $\mathbf{L}(u) = \partial_x(\sin(x)\partial_x u)$;

D. $\mathbf{L}(u) = \partial_{xx} u + u \partial_x u$

E. $\mathbf{L}(u) = u(x, 0)$

F. $\mathbf{L}(u) = c_1 u(0, t) + c_2 \partial_x u(1, t)$

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Linear homogeneous equation $\mathbf{L}(u) = f$ with $f = 0$
otherwise (if $f \neq 0$), nonhomogeneous.

- linearity and homogeneity also apply to BC.

Principle of Superposition $\mathbf{L}$ linear,

$$\text{if } \mathbf{L}(u_1) = \mathbf{L}(u_2) = 0, \text{ then } \mathbf{L}(c_1u_1 + c_2u_2) = 0 \; .$$

- if u_1, u_2 solve $\mathbf{L}(u) = 0$, then so does $c_1u_1 + c_2u_2$
- T/F? $\mathbf{L}(u_1) = f_1, \mathbf{L}(u_2) = f_2 \Rightarrow \mathbf{L}(u_1 + u_2) = f_1 + f_2$.

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Application: Solution Decomposition. Decompose the solution of

$$\begin{cases} \partial_t u = \kappa \partial_{xx} u + Q(x,t), & \text{with } x \in (0,L), t > 0\,; \\ \text{IC: } u(x,0) = f(x) & \text{with } x \in [0,L]\,, \\ \text{BC: } u(0,t) = \phi(t), u(L,t) = \psi(t) & \text{with } t > 0\,. \end{cases}$$

to $u(x,t) = v(x,t) + w(x,t)$ such that

$$\begin{cases} \partial_t v = \kappa \partial_{xx} v, \\ \text{IC: } v(x,0) = f(x)\,, \\ \text{BC: } v(0,t) = 0, v(L,t) = 0\,. \end{cases}$$

$$\begin{cases} \partial_t w = \kappa \partial_{xx} w + Q(x,t), \\ \text{IC: } w(x,0) = 0, \\ \text{BC: } w(0,t) = \phi(t), u(L,t) = \psi(t)\,. \end{cases}$$

HomoEq+ HomoBC;

HomoIC

Application 2. Consider

$$\begin{cases} \partial_t u = \kappa \partial_{xx} u, & \text{with } x \in (0,L), t > 0\,; \\ \text{IC: } u(x,0) = f(x) & \text{with } x \in [0,L]\,, \\ \text{BC: } u(0,t) = A, u(L,t) = B & \text{with } t > 0\,. \end{cases}$$

The displacement trick:

- Equilibrium solution: $u_E(x) = A + \frac{x}{L}(B - A)$.
- **Displacement from the equilibrium**: $v(x,t) = u(x,t) - u_E(x)$.
- We get

$$\begin{cases} \partial_t v = \kappa \partial_{xx} v, & \text{with } x \in (0,L), t > 0\,; \\ \text{IC: } v(x,0) = f(x) - u_E(x) & \text{with } x \in [0,L]\,, \\ \text{BC: } v(0,t) = 0, v(L,t) = 0 & \text{with } t > 0\,. \end{cases}$$

We will discuss the general Non-homogeneous case in Chapter 8.

Outline

HE: homogeneous IBVP

$$\partial_t u = \kappa \partial_{xx} u,$$
$$u(x,0) = f(x)$$
$$u(0,t) = 0, u(L,t) = 0$$

- equation and BC: linear homogeneous
- physical meaning:
  1D rod with no sources and both ends immersed at 0^o.
  How the temperature evolve to Equilibrium?
- a first step for general IBVP (from previous slide)
  can be solved by **method of separation of variables** $\downarrow$

Separation of variables

Seek solutions in the form (Daniel Bernoulli 1700s)

$$u(x,t) = \phi(x)G(t)$$

Reduce PDE to ODEs:

Separation of variables

Seek solutions in the form (Daniel Bernoulli 1700s)

$$u(x,t) = \phi(x)G(t)$$

Reduce PDE to ODEs:

$$\partial_t u = \phi(x)G'(t) = \kappa \partial_{xx} u = \kappa \phi''(x)G(t)$$

$$\frac{G'(t)}{\kappa G(t)} = \frac{\phi''(x)}{\phi(x)} \overset{\text{for any x,t}}{=} -\lambda$$

- λ is a constant TBD
- two ODEs:
  In time: $G'(t) = -\lambda\kappa G(t) \quad \Rightarrow$
  In space: $\phi''(x) = -\lambda\phi(x) \quad \Rightarrow$
- IC: trivial solution when $f(x) = 0$, $u \equiv 0$ with $G \equiv 0$;
  otherwise, $u(x,0) = G(0)\phi(x) = f(x)$: $G(0)$ TBD
- BC: for non-trivial solution $\Rightarrow \phi(0) = \phi(L) = 0$

Time dependent ODE

$$G'(t) = -\lambda\kappa G(t) \quad \Rightarrow G(t) = G(0)e^{-\lambda\kappa t}.$$

Assume that $G(0) > 0$,

- $\lambda < 0$: $G(t) \uparrow \infty$
- $\lambda = 0$:
- $\lambda > 0$:

Physical setting: $\lambda \geq 0$

Boundary value problem

$$\phi''(x) = -\lambda\phi(x), \quad \phi(0) = \phi(L) = 0$$

- $\lambda < 0$: $\phi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$
- $\lambda = 0$: $\phi(x) =$
- $\lambda > 0$: $\phi(x) =$

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Eigenfunctions: $\mathbf{L}\phi = \lambda\phi, \phi(0) = \phi(L) = 0$, with $\mathbf{L}\phi := -\phi''$

$$\phi_n(x) = \sin(\frac{n\pi}{L}x), \quad \lambda_n = (\frac{n\pi}{L})^2, \quad n = 1, 2, \cdots,$$

Solution to HE-IBVP:

$$\partial_t u = \kappa \partial_{xx} u,$$
$$u(x,0) = f(x)$$
$$u(0,t) = 0, u(L,t) = 0$$

$$\lambda_n = (\tfrac{n\pi}{L})^2, n = 1, 2, \ldots$$

$$u(x,t) = \phi_n(x) G_n(t) = \sin(\frac{n\pi}{L}x) e^{-\lambda_n \kappa t}$$

Solution to HE-IBVP:

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$$u(x,t) = \phi_n(x) G_n(t) = \sin(\frac{n\pi}{L} x) e^{-\lambda_n \kappa t}$$

PoS:

$$u_N(x,t) = \sum_{n=1}^{N} B_n \sin(\frac{n\pi}{L} x) e^{-\lambda_n \kappa t} \rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L} x) e^{-\lambda_n \kappa t}$$

- if $f(x) = \sum_{n=1}^{N} B_n \sin(\frac{n\pi}{L} x)$, u_N is a solution
- if $f(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L} x)$, u is a solution
  (convergence of function series: Chp3:Fourier series)

For a general f, how to determine B_n?

Solution to HE-IBVP:

$$\partial_t u = \kappa \partial_{xx} u,$$
$$u(x,0) = f(x)$$
$$u(0,t) = 0, u(L,t) = 0$$

$$\lambda_n = (\tfrac{n\pi}{L})^2, n = 1, 2, \ldots$$

$$u(x,t) = \phi_n(x) G_n(t) = \sin(\frac{n\pi}{L}x) e^{-\lambda_n \kappa t}$$

PoS:

$$u_N(x,t) = \sum_{n=1}^{N} B_n \sin(\frac{n\pi}{L}x) e^{-\lambda_n \kappa t} \to u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L}x) e^{-\lambda_n \kappa t}$$

- if $f(x) = \sum_{n=1}^{N} B_n \sin(\frac{n\pi}{L}x)$, u_N is a solution
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  (convergence of function series: Chp3:Fourier series)

For a general f, how to determine B_n? **Orthogonality**

$$\int_0^L \sin(\frac{n\pi}{L}x) \sin(\frac{m\pi}{L}x) dx = \delta_{m-n} \frac{L}{2}$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi}{L}x) dx$$

Compute B_m Multiply both sides by $\sin(\frac{m\pi}{L}x)$, and integrate them

$$\int_0^L f(x)\sin(\frac{m\pi}{L}x)dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin(\frac{n\pi}{L}x)\sin(\frac{m\pi}{L}x)dx$$
$$= B_m \int_0^L \sin^2(\frac{m\pi}{L}x)dx = \frac{B_m L}{2}\,.$$

(when can we exchange $\sum_{n=1}^{\infty}$ and $\int_0^L$?)

$$B_m = \frac{2}{L}\int_0^L f(x)\sin(\frac{m\pi}{L}x)dx\,.$$

Example: $f(x) \equiv 100$,

$$B_n = \frac{2}{L}\int_0^L 100\sin(\frac{n\pi}{L}x)dx = \frac{200}{L}\left(-\frac{L}{n\pi}\cos(\frac{n\pi}{L}x)\right)\Big|_0^L$$
$$= \frac{200}{n\pi}(1-\cos(n\pi)) = \begin{cases} 0 & n \text{ even}\,; \\ \frac{400}{n\pi} & n \text{ odd}\,. \end{cases}$$

Review of the method: separation of variables (SoV)

$$\underbrace{PDE}_{\text{linear, homo}} + \underbrace{BC}_{\text{linear, homo}} + IC$$

$$\partial_t u = \kappa \partial_{xx} u,$$
$$u(0,t) = 0, u(L,t) = 0$$
$$u(x,0) = f(x)$$

1. linear + homo $\Rightarrow$ PoS

2. SoV: PDE+BC $\Rightarrow$ ODEs

$$\frac{G'(t)}{\kappa G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

3. Solve EigenvalueP

$$G(t) = G(0)e^{-\lambda\kappa t}.$$

$$\phi''(x) = -\lambda\phi(x), \quad \phi(0) = \phi(L) = 0$$

4. IC $\Rightarrow$ coefficients

(orthogonality $\downarrow$)

$$\phi_n(x) = \sin(\frac{n\pi}{L}x), \ \lambda_n = (\frac{n\pi}{L})^2, n \geq 1$$

5. Conclude solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L}x)e^{-\lambda_n\kappa t}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx$$

Orthogonality

In finite dimensional space: $\mathbf{a} = (a_1, a_2, \ldots, a_N), \mathbf{b} \in \mathbb{R}^N$:

$$\mathbf{a} \perp \mathbf{b} \Leftrightarrow \langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{N} a_i b_i = 0$$

For functions: $\phi, \psi \in C[0, L]$ (connection?)

$$\phi \perp \psi \Leftrightarrow \langle \phi, \psi \rangle = \int_0^L \phi(x)\psi(x)dx = 0$$

Recall $\{\phi_n, \lambda_n\}$ with $\phi_n(x) = \sin(\frac{n\pi}{L}x)$ and $\lambda_n = \frac{n\pi}{L}$ solve:

$$\phi''(x) = -\lambda\phi(x), \quad \phi(0) = \phi(L) = 0$$

We have $\langle \phi_n, \phi_m \rangle = \delta_{m-n} \frac{L}{2}$.

Outline

Section 2.4: HE with other boundary values

HE+ $BC_{\text{Neumann, homo}}$ + IC

$$\partial_t u = \kappa \partial_{xx} u,$$
$$\partial_x u(0,t) = 0, \partial_x u(L,t) = 0$$
$$u(x,0) = f(x)$$

1. linear homo: $\Rightarrow$ PoS
2. SoV: $u(x,t) = \phi(x)G(t)$
3. Solve EigenvalueP
4. Determine coefs. by IC/BC.
5. Conclude solution

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n \kappa t} \phi_n(x), \quad \phi_n(x) = \cos(\frac{n\pi}{L} x)$$

$\lim_{t\to\infty} u(x,t) =?$

From the IC $u(x,0) = f(x)$ and the **orthogonality** relation:

$$\int_0^L \cos(\frac{n\pi}{L}x)\cos(\frac{m\pi}{L}x)dx := \begin{cases} 0 & m \neq n\,; \\ L/2 & m = n \neq 0\,; \\ L & m = n = 0\,, \end{cases}$$

we have for A_m (assuming exchange of $\sum_{n=1}^{\infty}$ and $\int_0^L$)

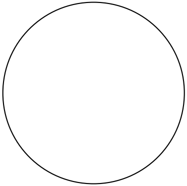
$$A_0 = \frac{1}{L}\int_0^L f(x)dx\,, \quad A_m = \frac{2}{L}\int_0^L f(x)\cos(\frac{m\pi}{L}x)dx\,, m = 1, 2, \cdots\,.$$

Then as $t \to \infty$, the solution approaches a steady state:

$$\lim_{t\to\infty} u(x,t) = A_0 = \frac{1}{L}\int_0^L f(x)dx\,.$$

HE in a circular ring

$$\partial_t u = \kappa \partial_{xx} u,$$
$$u(L,t) = u(-L,t)$$
$$\partial_x u(L,t) = \partial_x u(-L,t)$$
$$u(x,0) = f(x)$$



1. linear homo: $\Rightarrow$ PoS
2. SoV: $u(x,t) = \phi(x)G(t)$
3. Solve EigenvalueP
4. Determine coefs. by IC/BC.
5. Conclude solution

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{-\lambda_n \kappa t}[a_n \phi_n(x) + b_n \psi_n(x)]$$

$\lim_{t\to\infty} u(x,t) = ?$

Summary of boundary value problems for $\phi'' = -\lambda\phi$:

BOUNDARY VALUE PROBLEMS

Boundary conditions	$\phi(0)=0$ $\phi(L)=0$	$\frac{d\phi}{dx}(0)=0$ $\frac{d\phi}{dx}(L)=0$	$\phi(-L)=\phi(L)$ $\frac{d\phi}{dx}(-L)=\frac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\frac{n\pi}{L}\right)^2$ $n=1, 2, 3, \ldots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0, 1, 2, 3, \ldots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0, 1, 2, 3, \ldots$
Eigenfunctions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$
Series	$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L}$ $+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$
Coefficients	$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$	$A_0 = \frac{1}{L}\int_0^L f(x)\,dx$ $A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$	$a_0 = \frac{1}{2L}\int_{-L}^L f(x)\,dx$ $a_n = \frac{1}{L}\int_{-L}^L f(x)\cos\frac{n\pi x}{L}\,dx$ $b_n = \frac{1}{L}\int_{-L}^L f(x)\sin\frac{n\pi x}{L}\,dx$