Chapter 10: Fourier transform

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10.2 Heat equation on an infinite domain

10.3 Fourier transform pair

10.4 Fourier transform and heat equation

10.5: Fourier sine and cosine transforms

10.6 Examples using Fourier transform

Outline

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10.2 Heat equation on an infinite domain

Previous: PDE on bounded region (1D rod or 2D rectangular/circle).

What if **infinite regions**? What is the "boundary" condition?

- Influence at boundary is negligible
- Bounded total energy: $\int |u(x,t)|dx < \infty$
- Equilibrium?

1D infinite rod:

$$\partial_t u = \partial_{xx} u, \quad x \in (-\infty, \infty)$$
$$u(x,0) = f(x), \quad x \in (-\infty, \infty)$$
$$\text{BC: } u(x,t) \to 0 \text{ as } |x| \to \infty$$

10.2 Heat equation on an infinite domain

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BC: $u(x,t) \to 0$ as $|x| \to \infty$

Linear + Homogeneous
$\to$ separation of variables

- seek $u(x,t) = h(t)\phi(x) \to$
- superposition?
  $u(x,t) = \sum_{n=1}^{\infty} h_n(0) e^{-\lambda_n t} \phi_n(x)$

Eigenvalue problem

$$\phi'' = -\lambda\phi, \quad x \in (-\infty, \infty)$$
$\phi(x) \to 0$ as $|x| \to \infty$?

$\lambda > 0$:

$\lambda = 0$:

$\lambda < 0$:

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$\lambda > 0 :$

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$\lambda < 0 :$

ϕ bounded?

10.2 Heat equation on an infinite domain

We have eigenfunction $\phi_\lambda(x) = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x$ for every $\lambda \geq 0$. Uncountably many!

Fourier series:
$u(x,t) = \sum_{n=1}^{\infty} h_n(0) e^{-\lambda_n t} \phi_n(x)$
$h_n(0)$ from IC

Orthogonality
Completeness

Now: ?? $u(x,t) = \sum_{\lambda \geq 0} h_\lambda(0) e^{-\lambda t} \phi_\lambda(x)$??

$$= \int_0^\infty [c_1(\lambda) \sin\sqrt{\lambda}x + c_2(\lambda) \cos\sqrt{\lambda}x] e^{-\lambda t} d\lambda$$

$$= \int_0^\infty [A(\omega) e^{i\omega x} + B(\omega) e^{-i\omega x}] e^{-\omega^2 t} d\omega$$

$A(\omega)$ and $B(\omega)$ from IC?
$\phi_\lambda(x)$: Orthogonality, Completeness?

Complex expoential

$$u(x,t) = \int_{-\infty}^{\infty} e^{-\omega^2 t} c(\omega) e^{-i\omega x} d\omega$$
$$f(x) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} d\omega$$

$c(\omega)$ from f?

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10.3 Fourier transform pair

To determine $c(\omega)$: (from Fourier series to Fourier transform)
Recall Fourier series on $[-L, L]$:

$$(\lambda_n, \phi_n) = \left((\frac{n\pi}{L})^2, e^{-i\frac{n\pi}{L}x} \right)$$

$$\frac{f(x_+) + f(x_-)}{2} = \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{n\pi}{L}x}$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(y) e^{i\frac{n\pi}{L}y} dy$$

10.3 Fourier transform pair

To determine $c(\omega)$: (from Fourier series to Fourier transform)

Recall Fourier series on $[-L, L]$:

$$(\lambda, \phi_\lambda) = \left(\lambda, e^{\mp i\sqrt{\lambda}x}\right)$$

$$(\lambda_n, \phi_n) = \left((\frac{n\pi}{L})^2, e^{-i\frac{n\pi}{L}x}\right)$$

$$w_n = \frac{2n\pi}{2L} = \frac{n\pi}{L},\ c_n = \frac{\Delta\omega}{2\pi}\int_{-L}^{L} f(y)e^{i\omega_n y}dy$$

$$\frac{f(x_+) + f(x_-)}{2} = \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{n\pi}{L}x}$$

$$\Rightarrow \frac{\Delta\omega}{2\pi}\sum_{n=-\infty}^{\infty}(\frac{1}{2L}\int_{-L}^{L} f(y)e^{i\omega_n y}dy)e^{-i\omega_n x}$$

$\Delta\omega \to 0,\ L \to \infty$:

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(y)e^{i\frac{n\pi}{L}y}dy$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} f(y)e^{i\omega y}dy)e^{-i\omega x}d\omega$$

$$\frac{f(x_+) + f(x_-)}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} f(y)e^{i\omega y}dy)e^{-i\omega x}d\omega$$

Fourier Transform

$$\widehat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx = \mathcal{F}[f](\omega)$$

Inverse Fourier Transform

$$\frac{f(x_+) + f(x_-)}{2} = \int_{-\infty}^{\infty} \widehat{f}(\omega)e^{-i\omega x}d\omega$$

- the coefficient $1/(2\pi)$ is a conventional notion

Example: Fourier transforms of Gaussian

$$\mathcal{F}^{-1}[e^{-\alpha\omega^2}](x) = \sqrt{\frac{\pi}{\alpha}}e^{-\frac{x^2}{4\alpha}}$$

$$\mathcal{F}[e^{-\beta x^2}](\omega) = \sqrt{\frac{1}{4\pi\beta}}e^{-\frac{\omega^2}{4\beta}}$$

Proof:

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$$\partial_t u = \partial_{xx} u, \quad x \in (-\infty, \infty)$$
$$u(x,0) = f(x), \quad x \in (-\infty, \infty)$$
BC: $u(x,t) \to 0$ as $|x| \to \infty$

Apply Fourier transform

$$\mathcal{F}[\partial_t u] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \partial_t u(x,t) e^{i\omega x} dx$$
$$\mathcal{F}[\partial_{xx} u] = ?$$

$$\mathcal{F}[\partial_t u] = \mathcal{F}[\partial_{xx} u] \Leftrightarrow \partial_t \widehat{u} = -\omega^2 \widehat{u}$$

Gaussian kernel:

$$u(x,t) = \mathcal{F}^{-1}[\widehat{u}(\omega,t)](x) = \ldots = \qquad = \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} dy$$

- Influence function: $u(x,t)$ depends on the entire IC
- $t \to 0$: $\delta(x-y)$

Fundamental solution of the heat equation: $u(x,0) = \delta(x)$

$$u(x,t) =$$

Convolution Theorem:

$$\mathcal{F}(f * g) = 2\pi \widehat{f}\,\widehat{g}; \qquad \frac{1}{2\pi} f * g = \mathcal{F}^{-1}[\widehat{f}\,\widehat{g}]$$

- Proof: by definition
- In the solution to (HE): (with $G(t, x) = \frac{1}{2\pi} G_t(x)$)

$$u(x, t) = \frac{1}{2\pi}[f * G_t](x)$$
$$\widehat{u}(\omega, t) = \widehat{f}(\omega)\widehat{G}_t(\omega)$$

Convolution Theorem:

$$\mathcal{F}(f * g) = 2\pi \widehat{f}\,\widehat{g}; \qquad \frac{1}{2\pi} f * g = \mathcal{F}^{-1}[\widehat{f}\,\widehat{g}]$$

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$$u(x,t) = \frac{1}{2\pi}[f * G_t](x)$$
$$\widehat{u}(\omega,t) = \widehat{f}(\omega)\widehat{G}_t(\omega)$$

Parseval's Equality

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} \widehat{g}(\omega)\overline{\widehat{g}(\omega)}dx$$

- The Fourier series version?
- Proof: $\frac{1}{2\pi} f * g(0) = \mathcal{F}^{-1}[\widehat{f}\,\widehat{g}](0)$ and use that $\overline{\widehat{f}} = \widehat{\overline{f(-x)}}$.

Summary: using Fourier transform to solve heat equation

$$\partial_t u = \partial_{xx} u, \quad x \in (-\infty, \infty)$$
$$u(x,0) = f(x), \quad x \in (-\infty, \infty)$$
$$\text{BC: } u(x,t) \to 0, \partial_x u(x,t) \to 0 \text{ as } |x| \to \infty$$

Apply Fourier transform

$$\mathcal{F}[\partial_t u] = \partial_t \widehat{u};$$
$$\mathcal{F}[\partial_x v] = -i\omega \widehat{v};$$
$$\mathcal{F}^{-1}[e^{-\alpha\omega^2}](x) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{x^2}{4\alpha}}$$

$$\mathcal{F}[\partial_t u] = \mathcal{F}[\partial_{xx} u] \Leftrightarrow \partial_t \widehat{u} = -\omega^2 \widehat{u}$$
$$\Rightarrow \qquad \widehat{u}(\omega, t) = \widehat{f}(\omega) e^{-\omega^2 t}$$

Denote $G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}}$

$$u(x,t) = \mathcal{F}^{-1}[\widehat{u}(\cdot,t)](x) = 2\pi \mathcal{F}^{-1}[\widehat{f}\widehat{G}(\cdot,t)](x) = [f * G(\cdot,t)](x)$$

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Motivating example

$$\partial_t u = \partial_{xx} u, \quad x \in (0, \infty)$$
$$u(x,0) = f(x), \quad x \in (0, \infty)$$
$$u(x,0) = 0;$$

Separation of variables:

- $u(x,t) \sim h(t)\phi(x)$
- eigenvalue problem

$$\phi'' = -\lambda\phi, \qquad x \in (0, \infty)$$
$$\phi(0) = 0;$$

Fourier Sine Transform

$$S[f](\omega) = \widehat{f^s}(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)dx$$

$$S^{-1}[\widehat{f^s}](x) = f(x) = \int_0^\infty \widehat{f^s}(\omega)\sin(\omega x)d\omega$$

Fourier Cosine Transform

$$C[f](\omega) = \widehat{f^c}(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\omega x)dx$$

$$C^{-1}[\widehat{f^c}](x) = f(x) = \int_0^\infty \widehat{f^c}(\omega)\cos(\omega x)d\omega$$

- Transform of derivatives

$$S[\partial_x f](\omega) = -\omega C[f]; \qquad C[\partial_x f](\omega) \quad = \frac{2}{\pi} f(0) - wS[f](\omega);$$

- applications: solve the (HE)

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10.6 Application examples

1. Wave equation

$$\partial_{tt} u = c^2 \partial_{xx} u, \quad x \in (-\infty, \infty)$$
$$u(x,0) = f(x), \partial_t u(x,0) = 0;$$

To solve it, apply Fourier transform

$$\partial_{tt} \widehat{u} = -c^2 w^2 \widehat{u}(t,w), \quad w \in (-\infty, \infty)$$
$$\widehat{u}(0,w) = \widehat{f}(w), \partial_t \widehat{u}(0,w) = 0;$$

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)]$$

2. Laplace equation on a semi-infinite strip

$$\partial_{xx}u + \partial_{yy}u = 0, \quad x \in (0,L), y > 0$$
$$u(0,y) = g_1(y); \quad u(L,y) = g_2(y);$$
$$u(x,0) = f(x); bounded$$

- 1. homogenize BC (u_2)

$$u = u_1 + u_2$$

- 2. Solve them:
  FT
  Eigenfunction expansion

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$$\partial_{xx}u + \partial_{yy}u = 0, \quad x \in (0, L), y > 0$$
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- 1. homogenize BC (u_2)

$$u = u_1 + u_2$$

- 2. Solve them:
  FT
  Eigenfunction expansion

$$\partial_{xx}u_1 + \partial_{yy}u_1 = 0, \quad x \in (0, L)$$
$$u_1(0, y) = g_1(y); \quad u_1(L, y) = g_2(y);$$
$$u(x, 0) = 0; bounded$$

FT in y

$$\partial_{xx}u_2 + \partial_{yy}u_2 = 0, \quad x \in (0, L)$$
$$u_2(0, y) = 0; \quad u_2(L, y) = 0;$$
$$u_2(x, 0) = f(x); bounded$$

Fourier series in x.

3-4. Laplace equation on half-plane or a quarter-plane

Reading.

5. HE on the plane

$$\partial_t u = \partial_{xx} u + \partial_{yy} u, \quad x, y \in (-\infty, \infty)$$
$$u(x, y, 0) = f(x, y)$$

- the BCs by default
- Double Fourier transform
- solution

$$u(x, y, t) = \int_{-\infty}^{\infty} A(w_1, w_2) e^{-i(w_1 x + w_2 y)} e^{-(w_1^2 + w_2^2)t} dw_1 dw_2$$

Homework 10:

Exe 10.6.11: Eq.(1) on first quarter plane with BC

$$\begin{aligned}
&\partial_t u = \partial_{xx} u + \partial_{yy} u, \quad x > 0, y > 0 \\
&u(x, y, 0) = f(x, y) \\
&u(0, y, t) = 0, \partial_y u(x, 0, t) = 0
\end{aligned}$$

Hint: make extensions of f to the entire plane. The extensions (even/odd/mixed) should satisfy the BC.