Chapter 5: Sturm-Liouville Eigenvalue Problem

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Outline

Section 5.10: Approximation properties

Recall that in solution (HE) IBVP:

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \phi_n(x),$$

- $\{(\lambda_n, \phi_n)\}$: eigen-pairs to the Sturm-Liouville eigenvalue problem
- a_n from IC $f = \sum_{n=1}^{\infty} a_n e^{-\lambda_n 0} \phi_n(x)$.

Question: In computational practice, we can only use finitely many terms,

$$f \approx f_N(x) = \sum_{n=1}^{N} \alpha_n \phi_n(x)$$

Should we use $\{\alpha_n = a_n\}$? Is $\{(a_n, \phi_n)\}$ the best?

Example: $f(x) = e^x$. Which one to use?

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$e^x = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

Metric on function space

Metric: better in what sense?

$$\text{distance}(f, f_N)$$

$f(x)$ with $x \in [0, L]$, and f is piecewise smooth.

- Maximum/uniform error

$$\|f - f_N\|_\infty = \max_{x\in[0,L]} |f(x) - f_N(x)|$$

- root Mean square error (MSE)

$$\|f - f_N\|_\sigma = (\int_0^L |f(x) - f_N(x)|^2 \sigma(x) dx)^{1/2}$$

Optimal mean square approximation

Theorem. f_N with $\{\alpha_n = a_n\}$ achieves minimal MSE. That is,

$$a_{1:N} = \arg\min_{\alpha_{1:N}} \|f - \sum_{n=1}^{N} \alpha_n \phi_n\|_\sigma^2.$$

Furthermore, the minimizer is unique.
Proof: (Hint: $\mathcal{E}(\alpha) = \|f - \sum_{n=1}^{N} \alpha_n \phi_n\|_\sigma^2$)

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- Mean square error

$$E = \|f - f_N\|_\sigma^2 =$$

- Bessel's inequality

$$\|f\|_\sigma^2 \geq \|f_N\|_\sigma^2$$

- Parseval's equality

$$\|f\|_\sigma^2 = \sum_{n=1}^{\infty} a_n^2 \langle \phi_n, \phi_n \rangle_\sigma$$

Example and exe

5.10.5. Show that if

$$L(f) = \frac{d}{dx}\left(p\frac{df}{dx}\right) + qf,$$

then

$$-\int_a^b fL(f)\,dx = -pf\frac{df}{dx}\bigg|_a^b + \int_a^b \left[p\left(\frac{df}{dx}\right)^2 - qf^2\right]\,dx$$

if f and df/dx are continuous.

5.10.6. Assuming that the operations of summation and integration can be interchanged, show that if

$$f = \sum \alpha_n \phi_n \quad \text{and} \quad g = \sum \beta_n \phi_n,$$

then for normalized eigenfunctions

$$\int_a^b fg\sigma\,dx = \sum_{n=1}^{\infty} \alpha_n \beta_n,$$

a generalization of Parseval's equality.

5.10.7. Using Exercises 5.10.5 and 5.10.6, prove that

$$-\sum_{n=1}^{\infty} \lambda_n \alpha_n^2 = -pf\frac{df}{dx}\bigg|_a^b + \int_a^b \left[p\left(\frac{df}{dx}\right)^2 - qf^2\right]\,dx. \qquad (5.10.15)$$

[*Hint*: Let $g = L(f)$, assuming that term-by-term differentiation is justified.]

Outline

Section 5.10: Approximation properties

Section 5.9: Large eigenvalues (Asymptotical behavior)

How to estimate λ_n as $\rightarrow \infty$?

$$(p\phi')' + q\phi = \lambda\sigma(x)\phi$$

When λ is large, two eigenfunctions (sine and cosine) are close to

$$\phi(x) \approx (\sigma p)^{-1/4} e^{\pm i\lambda^{1/2} \int^x (\frac{\sigma}{p})^{1/2} dx_0}$$

Then, we can estimate λ by applying the boundary value:

$$\lambda_n \approx \left(\frac{n\pi}{C_L}\right)^2, \quad C_L = \int_0^L (\frac{\sigma(x_0)}{p(x_0)})^2 dx_0$$

- Main idea: local approximation
- In the "derivation": λ large, $\lambda\sigma(x_0) >> q(x)$ in $O(x_0)$.
  $\rightarrow$ consider a local version: $(p(x_0)\phi')' = \lambda\sigma(x_0)\phi$