

Chapter 7: High dimensional PDEs

Fei Lu

Department of Mathematics, Johns Hopkins

7.2: Separation of time variable d-D

7.3 Vibrating Rectangular Membrane

7.4 The eigenvalue problem

7.5: Green's formula

7.6: Rayleigh Quotient and Laplace's Equation

7.7: Vibrating Circular Membrane

Outline

7.2: Separation of time variable d-D

7.3 Vibrating Rectangular Membrane

7.4 The eigenvalue problem

7.5: Green's formula

7.6: Rayleigh Quotient and Laplace's Equation

7.7: Vibrating Circular Membrane

Section 7.2: Separation of time variable d-D

We have solved 1D equations. Will everything work for 2D& 3D?

1D: $x \in [0, L]$

HE $\partial_t u = \partial_{xx} u$

WE $\partial_{tt} u = \partial_{xx} u$

BC & IC

$$u(t, x) = h(t)\phi(x) : \downarrow$$

$$\phi''(x) = -\lambda\phi; \quad \text{BC}$$

$$h'(t) \text{ or } h''(t) = -\lambda_n h, \quad \text{IC}$$

$$\Rightarrow (\lambda_n, \phi_n), h_n$$

$$u(t, x) = \sum_{n=1}^{\infty} h_n(t)\phi_n(x)$$

Sturm-Liouville:

$$(p\phi')' + q\phi = -\lambda\sigma\phi$$

Method of eigenfunction expansion

$$\partial_{xx} u \rightarrow ?$$

$$\mathbf{2D:} \nabla^2 u = (\partial_{xx} + \partial_{yy})u$$

$$\mathbf{3D:} \nabla^2 u = (\partial_{xx} + \partial_{yy} + \partial_{zz})u$$

$$\mathbf{d-D:} \nabla^2 u = \sum_{i=1}^d \partial_{x_i x_i} u$$

Second order PDEs

HE $\partial_t u = \nabla^2 u$

WE $\partial_{tt} u = \nabla^2 u$

BC: $[A(t)u + B(t)\nabla u \cdot \mathbf{n}]|_{\partial\Omega} = 0$

IC: $u(x, 0), \partial_t u(x, 0)$

Separate time variable

Consider (WE)

(HE): Similar

$$\partial_{tt}u = c^2(\partial_{xx}u + \partial_{yy}u)$$

$$u(x, y, 0) = \alpha(x, y); \partial_t u(x, y, 0) = \beta(x, y);$$

$$\text{BC: } [A(t)u + B(t)\nabla u \cdot \mathbf{n}]|_{\partial\Omega} = 0;$$

Seek $u(x, y, t) = h(t)\phi(x, y)$:
from the equation, we have

$$h''(t)\phi = c^2(\partial_{xx}\phi + \partial_{yy}\phi)h$$

$$\frac{h''}{c^2 h} = \frac{\partial_{xx}\phi + \partial_{yy}\phi}{\phi} = -\lambda$$

How do we solve

$$\partial_{xx}\phi + \partial_{yy}\phi = -\lambda\phi?$$

Recall Laplace equ. in Chp 2.

Outline

7.2: Separation of time variable d-D

7.3 Vibrating Rectangular Membrane

7.4 The eigenvalue problem

7.5: Green's formula

7.6: Rayleigh Quotient and Laplace's Equation

7.7: Vibrating Circular Membrane

7.3 Vibrating Rectangular Membrane

BC: $x \in [0, L]; y \in [0, H],$

$$u(0, y, t) = 0 = u(L, y, t);$$

$$u(x, 0, t) = 0 = u(x, H, t);$$

Simulation video: 5:50

Eigenvalue problem

$$\partial_{xx}\phi + \partial_{yy}\phi = -\lambda\phi;$$

$$\phi(0, y) = 0 = \phi(L, y);$$

$$\phi(x, 0) = 0 = \phi(x, H); \nearrow$$

Separation of variables (again)

$$\phi(x, y) = f(x)g(y)$$

$$\frac{f''}{f} = -\lambda - \frac{g''}{g} = -\mu$$

$$\lambda_{n,m} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2,$$
$$\phi_{n,m}(x, y) = \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{H}y\right)$$

Back to the equation,

$$\partial_{tt}u = c^2(\partial_{xx}u + \partial_{yy}u)$$

$$\text{IC: } u(x, y, 0) = \alpha(x, y); \partial_t u(x, y, 0) = \beta(x, y);$$

$$\text{BC: } \dots$$

$$\text{SoV: } u(x, t) = h_{n,m}(t)\phi_{n,m}(x, y)$$

To determine $h_{n,m}(t)$:

$$h''(t) = -\lambda_{n,m}c^2h,$$

$$\text{IC } h(0), h'(0) = ?$$

$$h_{n,m}(t) = a_{n,m} \cos(c\sqrt{\lambda_{n,m}}t) + b_{n,m} \sin(c\sqrt{\lambda_{n,m}}t)$$

Back to the equation,

$$\partial_{tt}u = c^2(\partial_{xx}u + \partial_{yy}u)$$

$$\text{IC: } u(x, y, 0) = \alpha(x, y); \partial_t u(x, y, 0) = \beta(x, y);$$

BC: ...

$$\text{SoV: } u(x, t) = h_{n,m}(t)\phi_{n,m}(x, y)$$

To determine $h_{n,m}(t)$:

$$h''(t) = -\lambda_{n,m}c^2h,$$

$$\text{IC } h(0), h'(0) = ?$$

$$h_{n,m}(t) = a_{n,m} \cos(c\sqrt{\lambda_{n,m}}t) + b_{n,m} \sin(c\sqrt{\lambda_{n,m}}t) \quad \text{General solution}$$

$$(\text{MEE}): u(x, t) = \sum_{n,m=1}^{\infty} h_{n,m}(t)\phi_{n,m}(x, y)$$

Orthogonal? Complete?

Outline

7.2: Separation of time variable d-D

7.3 Vibrating Rectangular Membrane

7.4 The eigenvalue problem

7.5: Green's formula

7.6: Rayleigh Quotient and Laplace's Equation

7.7: Vibrating Circular Membrane

7.4 The eigenvalue problem

1D Sturm-Liouville Theorem

$$(p(x)\phi')' + q(x)\phi = -\lambda\sigma\phi$$

$$\beta_1\phi(a) + \beta_2\phi'(a) = 0;$$

$$\beta_3\phi(b) + \beta_4\phi'(b) = 0;$$

regular **SLEP** $\{(\lambda_n, \phi_n)\}$ s.t.

1-2 $\{\lambda_n \uparrow \infty\}_{n=1}^{\infty}$ strictly

3 ϕ_n is unique to λ_n ;
 ϕ_n has $n - 1$ zeros

4 $\{\phi_n\}_{n=1}^{\infty}$ is complete.

5 $\{\phi_n\}_{n=1}^{\infty}$ are orthogonal

6 Rayleigh quotient $\lambda_n = -\frac{\langle L\phi_n, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle_{\sigma}};$

2D& 3D Helmholtz Eq.

$$\nabla^2 \phi = -\lambda\sigma\phi$$

$$a\phi + b\nabla\phi \cdot \mathbf{n} |_{\partial\Omega} = 0;$$

$$\nabla \cdot (p\nabla\phi) + q\phi = -\lambda\sigma\phi$$

1-2 $\{\lambda_n \nearrow \infty\}_{n=1}^{\infty}$, may repeat

3 $\lambda_n = \lambda_{n+1}$: multiple eigen-fun

4 $\{\phi_n\}_{n=1}^{\infty}$ is complete.

5 $\{\phi_n\}_{n=1}^{\infty}$ are orthogonal
(Gram-schmidt orthogonalization)

6 Rayleigh quotient $\lambda_n = -\frac{\langle L\phi_n, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle_{\sigma}};$

Outline

7.2: Separation of time variable d-D

7.3 Vibrating Rectangular Membrane

7.4 The eigenvalue problem

7.5: Green's formula

7.6: Rayleigh Quotient and Laplace's Equation

7.7: Vibrating Circular Membrane

7.5: Green's formula

Orthogonality of eigenfunctions:

$$\begin{aligned} L\phi &= \nabla^2 \phi = -\lambda \phi && \text{in } \Omega \\ a\phi + b\nabla \phi \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \end{aligned}$$

Green's formula



$$\int_{\Omega} (uLv - vLu) dx = \oint_{\partial\Omega} (u\nabla v - v\nabla u) \cdot \mathbf{n} dS$$

Self-adjoint operator

$$L = L^*: \langle u, Lv \rangle = \langle L^*u, v \rangle$$

$$\int_{\Omega} (uLv - vLu) dx = 0$$

L is self-adjoint on the function spaces

- ▶ $C_0^2 = \{w : w \in C^2(\Omega), w|_{\partial\Omega} = 0\}$
- ▶ $C_I^2 = \{w : w \in C^2(\Omega), \nabla w \cdot \mathbf{n}|_{\partial\Omega} = 0\}$

Proof of the Green's formula:
Divergence theorem

$$\int_{\Omega} \nabla \mathbf{A} dx = \oint \mathbf{A} \cdot \mathbf{n} dS$$

Green's formula: application

Orthogonality of eigenfunctions L self-adjoint

Let $\{\lambda_i, \phi_i\}$ and $\{\lambda_j, \phi_j\}$ be two eigen-pairs, $\lambda_i \neq \lambda_j$. Then $\langle \phi_i, \phi_j \rangle = 0$.

Proof:

Question: How about ϕ_i and ϕ_j for $\lambda_i = \lambda_j$? Gram-Schmidt

Outline

7.2: Separation of time variable d-D

7.3 Vibrating Rectangular Membrane

7.4 The eigenvalue problem

7.5: Green's formula

7.6: Rayleigh Quotient and Laplace's Equation

7.7: Vibrating Circular Membrane

7.6: Rayleigh Quotient and Laplace's Equation

Let L be self-adjoint. If $L\phi = -\lambda\phi$, then $\lambda = -\frac{\langle L\phi, \phi \rangle}{\langle \phi, \phi \rangle}$

Application 1. $L = \nabla^2$. Suppose:

$$\begin{aligned}\nabla^2\phi &= -\lambda\phi && \text{in } \Omega \\ \phi &= 0 && \text{on } \partial\Omega,\end{aligned}$$

show that $\lambda > 0$.

($\Rightarrow$ all eigenvalues > 0)

If insulated BC: $\nabla u(t, \cdot) \cdot \mathbf{n} = 0$?

7.6: Rayleigh Quotient and Laplace's Equation

Let L be self-adjoint. If $L\phi = -\lambda\phi$, then $\lambda = -\frac{\langle L\phi, \phi \rangle}{\langle \phi, \phi \rangle}$

Application 2. Heat equation:

$$\partial_t u = \nabla^2 u \text{ in } \Omega$$

$$u(t, \cdot) = 0 \text{ on } \partial\Omega,$$

$$u(0, x) = f(x)$$

$$\int_{\Omega} u \nabla^2 u dx = - \int_{\Omega} |\nabla u|^2 dx + \oint u \nabla u \cdot \mathbf{n} dS$$

1. show that $\lim_{t \rightarrow \infty} u(t, x) = 0, \forall x$.

Hint: solution & $\lambda > 0$

2. If insulated BC: $\nabla u(t, \cdot) \cdot \mathbf{n} = 0$?

Outline

7.2: Separation of time variable d-D

7.3 Vibrating Rectangular Membrane

7.4 The eigenvalue problem

7.5: Green's formula

7.6: Rayleigh Quotient and Laplace's Equation

7.7: Vibrating Circular Membrane

7.7: Vibrating Circular Membrane

Consider $\Omega =$ disk with radius a

$$\partial_{tt}u = \nabla^2 u \text{ in } \Omega$$

$$u(a, \theta, t) = 0, \theta \in [-\pi, \pi]$$

$$u(r, \theta, 0) = \alpha(r, \theta), \partial_t u(r, \theta, t) = \beta(r, \theta)$$

Simulation video

Outline:

1. Separation of variables

$$u(r, \theta, t) = \phi(r, \theta)h(t)$$

$$h''(t)\phi = c^2(\nabla^2\phi)h$$

$$\frac{h''}{c^2h} = \frac{\nabla^2\phi}{\phi} = -\lambda$$

2. Solve the eigenvalue problem

$$\nabla^2\phi = -\lambda\phi \text{ \& BC}$$

► complete and orthogonal

3. Eigenfunction expansion

$$u(r, \theta, t) = \sum_{n=1}^{\infty} h_n(t)\phi_n(r, \theta)$$

7.7: Vibrating Circular Membrane

2. Solve the eigenvalue problem

$$\begin{aligned}\nabla^2 \phi &= -\lambda \phi, & r \in (0, a), \theta \in (-\pi, \pi) \\ \phi(a, \theta) &= 0, & \theta \in (-\pi, \pi)\end{aligned}$$

Seek $\phi(r, \theta) = f(r)g(\theta)$:

► Eigenvalue Problem A:

$$\begin{aligned}g''(\theta) &= -\mu g, & \theta \in (-\pi, \pi) \\ g(-\pi) &= g(\pi), g'(-\pi) = g'(\pi)\end{aligned}$$

► Eigenvalue Problem B:

$$\begin{aligned}r(rf')' + (\lambda r^2 - \mu)f &= 0, & r \in (0, a) \\ f(a) &= 0, f(0) \text{ bounded}\end{aligned}$$

7.7: Vibrating Circular Membrane

2. Solve the eigenvalue problem

$$\begin{aligned}\nabla^2 \phi &= -\lambda \phi, \quad r \in (0, a), \theta \in (-\pi, \pi) \\ \phi(a, \theta) &= 0, \quad \theta \in (-\pi, \pi)\end{aligned}$$

Seek $\phi(r, \theta) = f(r)g(\theta)$:

► Eigenvalue Problem A:

$$\begin{aligned}g''(\theta) &= -\mu g, \quad \theta \in (-\pi, \pi) \\ g(-\pi) &= g(\pi), g'(-\pi) = g'(\pi)\end{aligned}$$

► Eigenvalue Problem B:

$$\begin{aligned}r(rf')' + (\lambda r^2 - \mu)f &= 0, \quad r \in (0, a) \\ f(a) &= 0, f(0) \text{ bounded}\end{aligned}$$

Eigenvalue Problem A:

$$\begin{aligned}\mu_m &= m^2, m \geq 0; \\ g_m(\theta) &= \sin m\theta, \text{ or } \cos m\theta.\end{aligned}$$

Eigenvalue Problem B: SLEP?

$$(rf')' - \frac{m^2}{r}f = -\lambda rf, \quad r \in (0, a)$$

Still valid: singular SLEP.

Can $\lambda = 0$?

More about $\lambda_{n,m}$ and $\phi_{n,m}$?

Bessel functions

Eigenvalue Problem B:

$$r(rf')' + (\lambda r^2 - \mu)f = 0, \quad r \in (0, a)$$

$$f(a) = 0, f(0) \text{ bounded}$$

$$r^2 f'' + rf' + (\lambda r^2 - m^2)f = 0$$

A change of variables (with $\lambda > 0$)

$$z = \sqrt{\lambda}r, h(z) = f(z/\sqrt{r}) \rightarrow$$

$$z^2 h'' + zh' + (z^2 - m^2)h = 0$$

$z \in (0, 1)$: Bessel's DE of order m

First-kind ($\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$):

$$J_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+1+m)} \left(\frac{z}{2}\right)^{m+2k}$$

Bessel functions

Eigenvalue Problem B:

$$r(rf')' + (\lambda r^2 - \mu)f = 0, \quad r \in (0, a)$$

$$f(a) = 0, f(0) \text{ bounded}$$

$$r^2 f'' + rf' + (\lambda r^2 - m^2)f = 0$$

A change of variables (with $\lambda > 0$)

$$z = \sqrt{\lambda}r, h(z) = f(z/\sqrt{\lambda}) \rightarrow$$

$$z^2 h'' + zh' + (z^2 - m^2)h = 0$$

$z \in (0, 1)$: Bessel's DE of order m

First-kind ($\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$):

$$J_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+1+m)} \left(\frac{z}{2}\right)^{m+2k}$$

$$h'' + z^{-1}h' + (1 - m^2 z^{-2})h = 0$$

► singular point $z = 0$

Near $z = 0$:

► $m \neq 0$: $z^2 \ll m^2$
 h bounded $\rightarrow z^2 h \ll m^2 h$
 h', h'' can be large: keep them

$$z^2 h'' + zh' - m^2 h \approx 0$$

Solution $h(z) \approx z^m$ or z^{-m}

► $m = 0$: $h(z) \approx 1$ or $\ln z$

Bessel functions

First kind of order m (bounded)

$$J_m(z) \sim \begin{cases} 1, & m = 0 \\ \frac{1}{2^m m!} z^m, & m > 0 \end{cases}$$

2nd kind of order m (singular)

$$Y_m(z) \sim \begin{cases} \frac{2}{\pi} \ln z, & m = 0 \\ -\frac{2^m (m-1)!}{\pi} z^{-m}, & m > 0 \end{cases}$$

Eigenvalue Problem B:

$$f(r) = c_1 J_m(\sqrt{\lambda} r) + c_2 Y_m(\sqrt{\lambda} r)$$

- ▶ $f(0)$ bounded $\rightarrow c_2 = 0$.
- ▶ $f(a) = 0 \rightarrow$ determine $\lambda_{n,m} = (z_{n,m}/a)^2$: zeros of $J_m(\sqrt{\lambda} r)$.

Eigenfunctions: $J_m(\sqrt{\lambda_{n,m}} r)$; complete and orthogonal $\sigma(r) = r$

back to the Vibrating Circular Membrane

Consider $\Omega =$ disk with radius a

$$\partial_{tt}u = \nabla^2 u \text{ in } \Omega$$

$$u(a, \theta, t) = 0, \theta \in [-\phi, \phi]$$

$$u(r, \theta, 0) = \alpha(r, \theta), \partial_t u(r, \theta, t) = \beta(r, \theta)$$

1. Separation of variables $u(r, \theta, t) = \phi(r, \theta)h(t)$
2. Solve the eigenvalue problem: $\nabla^2 \phi = -\lambda \phi + \text{BC}$
 - ▶ complete and orthogonal
3. Eigenfunction expansion

$$u(r, \theta, t) = \sum_{n,m=1}^{\infty} h_{n,m}(t) \phi_{n,m}(r, \theta)$$

$$\phi_{n,m} = J_m(\sqrt{\lambda_{n,m}}r) \begin{pmatrix} \cos(m\theta) \\ \sin(m\theta) \end{pmatrix} \begin{pmatrix} \cos(c\sqrt{\lambda_{n,m}}t) \\ \sin(c\sqrt{\lambda_{n,m}}t) \end{pmatrix}$$