Chapter 5: Sturm-Liouville Eigenvalue Problem

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Solution to the IBVP?

$$c(x)\rho(x)\partial_t u = K_0 \partial_{xx} u + Q(x,t), \quad \text{with } x \in (0,L), t \geq 0$$

IC: $u(x,0) = f(x)$,

BC: $u(0,t) = \phi(t), u(L,t) = \psi(t)$

Outline

Review: Eigenvalue problems in PDE

Recall that for Heat Equation and Wave Equation,

HE $\partial_t u = \partial_{xx} u$
WE $\partial_{tt} u = \partial_{xx} u$
BC $u(a,t)$, $\partial_x u(a,t)$, /mixed
$(a = 0, L, -L)$

$\phi''(x) = -\lambda\phi$
BC:$\phi(a), \phi'(a), /mixed$

$\lambda_n = (\frac{n\pi}{L})^2$, $\phi_n(x) = \sin\sqrt{\lambda_n}x$, $\cos\sqrt{\lambda_n}x$, or both

Quiz: which did we use in eigenfunction expansion method?

- A ∞ many eigenvalues $\lambda_n \in \mathbb{R}$
- B $\{\phi_n\}$ are orthogonal
- C $\{\phi_n\}$ are complete (Fourier Theorem)
- D TBTD conditions on u

Review: Eigenvalue problems in PDE

Apply it for non-constant coefficient equations?

Heat Flow in a non-uniform rod $c(x)\rho(x)\partial_t u = \partial_x(K_0\partial_x u) + \alpha u$

symmetry heat flow $\partial_t u = k\frac{1}{r}\frac{\partial}{\partial_r}(r\partial_r u)$

Separation of variables $\rightarrow$ eigenvalue problems

$$u(x,t) = h(t)\phi(x) \quad \rightarrow \quad (K_0\phi')' + \alpha\phi = -\lambda c(x)\rho(x)\phi$$
$$u(r,t) = h(t)\phi(r) \quad \rightarrow \quad (r\phi')' = -\lambda\phi$$

BC matters!

$\rightarrow$ **Sturm-Liouville Eigenvalue Problems**(SLEP)

$$(p(x)\phi')' + q(x)\phi = -\lambda\sigma\phi$$
$$\beta_1\phi(a) + \beta_2\phi'(a) = 0;$$
$$\beta_3\phi(b) + \beta_4\phi'(b) = 0; \text{ other BC}$$

When does this SLEP has eigenfunctions orthogonal and complete?

Outline

Section 5.3: Sturm-Liouville Eigenvalue Problem

Regular SLEP:

$$(p(x)\phi')' + q(x)\phi = -\lambda\sigma\phi \qquad p', q, \sigma \in C[a,b],$$
$$\beta_1\phi(a) + \beta_2\phi'(a) = 0; \qquad p(x) > 0, \sigma(x) > 0, \forall x \in [a,b]$$
$$\beta_3\phi(b) + \beta_4\phi'(b) = 0; \qquad \beta_1^2 + \beta_2^2 > 0, \beta_3^2 + \beta_4^2 > 0,$$

Theorem (Sturm-Liouville Theorems)

A regular SLEP has eigenvalues and eigenfunctions $\{(\lambda_n, \phi_n)\}$ *s.t.*

1-2 $\{\lambda_n\}_{n=1}^{\infty}$ *are real and strictly increasing to* ∞

3 ϕ_n *is the unique (up to a *factor) solution to* λ_n*;* ϕ_n *has* $n-1$ *zeros*

4 $\{\phi_n\}_{n=1}^{\infty}$ *is complete. That is, any piecewise smooth* f *can be represented by a generalized Fourier series* $f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x) = \frac{1}{2}[f(x_-) + f(x_+)]$

5 $\{\phi_n\}_{n=1}^{\infty}$ *are orthogonal:* $\langle\phi_n, \phi_m\rangle_\sigma = 0$ *if* $n \neq m$*;* $\langle\phi_n, \phi_n\rangle_\sigma > 0$

6 *Rayleigh quotient* $\lambda_n = -\frac{\langle \mathbf{L}\phi_n, \phi_n\rangle}{\langle\phi_n, \phi_n\rangle_\sigma}$*;*

$\langle f, g\rangle := \int_a^b f(x)g(x)dx; \quad \langle f, g\rangle_\sigma := \int_a^b f(x)g(x)\sigma(x)dx$

Section 5.3: Sturm-Liouville Eigenvalue Problem

Regular SLEP:

$$(p(x)\phi')' + q(x)\phi = -\lambda\sigma\phi$$
$$\beta_1\phi(a) + \beta_2\phi'(a) = 0;$$
$$\beta_3\phi(b) + \beta_4\phi'(b) = 0;$$

$$p', q, \sigma \in C[a,b],$$
$$p(x) > 0, \sigma(x) > 0, \forall x \in [a,b]$$
$$\beta_1^2 + \beta_2^2 > 0, \beta_3^2 + \beta_4^2 > 0,$$

Suppose $x \in (0, L)$, we have studied SLEP $\phi'' = -\lambda\phi$: $p(x) \equiv 1, q(x) \equiv 0, \sigma(x) \equiv 1$. Are they regular?

- A $\phi(0) = \phi(L) = 0$
- B $\phi'(0) = \phi'(L) = 0$
- C $\phi(0) + \phi'(a) = 0$; $\phi(L) + \phi'(L) = 0$
- D $\phi(-L) = \phi(L)$, $\phi'(-L) = \phi'(L)$

Section 5.3: Sturm-Liouville Eigenvalue Problem

Suppose $x \in (0, L)$, we have studied SLEP $\phi'' = -\lambda\phi$:
$\lambda_n = (\frac{n\pi}{L})^2$, $\phi_n(x) = \sin\sqrt{\lambda_n}x$, $\cos\sqrt{\lambda_n}x$, or both, depending on BC.

Do we have the properties in the SL theorem?

1-2 $\{\lambda_n\}_{n=1}^{\infty}$ are real and increasing to ∞

3 ϕ_n is the unique (up to a *factor) solution to λ_n; ϕ_n has $n-1$ zeros

4 $\{\phi_n\}_{n=1}^{\infty}$ is complete

5 $\{\phi_n\}_{n=1}^{\infty}$ are orthogonal

6 Rayleigh quotient $\lambda_n = -\frac{\langle \mathbf{L}\phi_n, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle_\sigma}$

Transform to SLEP form

Example1 Change the equation to the form of SLEP

$$\phi''(x) + \alpha(x)\phi' + [\lambda\beta(x) + \gamma(x)]\phi = 0$$

Transform to SLEP form

Example2 Exe.5.3.9. Consider the BVP:

$$x^2\phi'' + x\phi' + \lambda\phi = 0, \quad x \in (1, b)$$
$$\phi(1) = 0; \phi(b) = 0$$

(a) Write the equation in the SLE form.
(b) Show that $\lambda \geq 0$ for all (λ, ϕ) that solves the BVP.
(c) Determine all positive eigenvalues. Is $\lambda = 0$ an eigenvalue?

Outline

Section 5.1-2* Introduction and motivation

Section 5.3: Sturm-Liouville Eigenvalue Problem

Section 5.4: Example: heat flow in a non-uniform rod

Extra: Numerical solution to SLEP

Heat flow in a non-uniform rod

$c(x)\rho(x)\partial_t u = K_0\partial_{xx}u,$ with $x \in (0,L), t > 0$

IC: $u(x,0) = f(x),$

BC: $u(0,t) = 0, \partial_x u(L,t) = 0$

Find the solution (by eigenfunction expansion):

- separation of variables $\rightarrow$ eigenvalue problem
  $u(x,t) = h(t)\phi(x) \rightarrow (K_0\phi')' = -\lambda c(x)\rho(x)\phi$
- Solve the IBVP.

- positive c, ρ, K_0
- left side $=0^o$, insulated right side

Heat flow in a non-uniform rod

$c(x)\rho(x)\partial_t u = K_0 \partial_{xx} u,$

IC: $u(x,0) = f(x),$

BC: $u(0,t) = 0, \partial_x u(L,t) = 0$

Solution

$u(x,t) = \sum_{i=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t}$.

What is the large time behavior? ($\lim_{t\to\infty} u(x,t)$?)

- if $\lambda_n > 0$ for all n
- if $\lambda_n = 0$ for some n

  can it happen?
- if $\lambda_n < 0$?

Heat flow in a non-uniform rod

$c(x)\rho(x)\partial_t u = K_0\partial_{xx}u,$

IC: $u(x,0) = f(x),$

BC: $u(0,t) = 0, \partial_x u(L,t) = 0$

Solution

$u(x,t) = \sum_{i=1}^{\infty} a_n\phi_n(x)e^{-\lambda_n t}$.

What does the solution look like? Numerical solution represent the solution at discrete space-time grids:

$$x \in [0,L] \to 0 = x_0 < x_1 < \cdots < x_{d+1} = L, \quad x_i = iL/d;$$
$$t \in [0,T] \to 0 = t_0 < t_1 < \cdots < t_{N+1} = T, \quad t_j = jT/N;$$

- Find (λ_n, ϕ_n); $(K_0\phi')' = -\lambda c(x)\rho(x)\phi \downarrow$
- Find a_n from IC
- $u(x_i, t_j)$

Outline

Numerical solution to SLEP

$$(K_0(x)\phi')' = -\lambda c(x)\rho(x)\phi, \quad \phi(0) = \phi(L) = 0;$$

$$\mathbf{L}\phi = \lambda\phi \text{ with BC} \quad \rightarrow \quad \mathbf{A}\mathbf{y} = \lambda\mathbf{y}$$

$$\text{SLEP} \quad \rightarrow \quad \text{Linear algebra eigenvalue problem}$$

What is $\mathbf{A}$? Function $\leftrightarrow$ vector?

$$x_i = i\Delta x, i = 0, \ldots, d+1. \rightarrow \quad \mathbf{y} = (\phi(x_1), \ldots, \phi(x_d))$$