Chapter 8: Non-homogeneous Equations

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Solution to the IBVP with source and non-homo BC?

$$\partial_t u = \kappa \partial_{xx} u + Q(x,t), \quad \text{with } x \in (0,L), t \geq 0$$
$$u(x,0) = f(x)$$
$$\text{BC: } u(0,t) = \phi(t), u(L,t) = \psi(t)$$

Section 8.2: Heat flow with source and non-homo BC
Section 8.3: Methods of eigenfunction expansion (homo-BC)
Section 8.4: MEE (non-homo BC): after Chp5
Section 8.5: Forced vibrating membrane and Resonance
Section 8.6: Poisson's Equition

Outline

Section 8.2: Heat flow with source and non-homo BC

1. Time-independent BC

Consider first

$$\partial_t u = \kappa \partial_{xx} u, \quad \text{with } x \in (0, L), t > 0$$
$$u(x, 0) = f(x)$$
$$\text{BC: } u(0, t) = A, u(L, t) = B$$

1> Equilibrium solu $u_E(x) = A + \frac{x}{L}(B - A)$.

Section 8.2: Heat flow with source and non-homo BC

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2> Displacement from Equilibrium

$$u = v + u_E : \quad v(x,t) = u(x,t) - u_E(x)$$

$\partial_t v = \kappa \partial_{xx} v,$

$v(x,0) = f(x) - u_E(x)$

$v(0,t) = 0, v(L,t) = 0$

$$v(x,t) = \sum_{n=0}^{\infty} a_n e^{-\kappa \lambda_n t} \sin \frac{n\pi}{L} x$$

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Extension (exe): steady source

$$\partial_t u = \kappa \partial_{xx} u + Q(x)$$
$$u(x, 0) = f(x)$$
$$u(0, t) = A, u(L, t) = B$$

2. Time-dependent non-homo PDE&BC

$$\begin{aligned} \partial_t u =& \ \kappa \partial_{xx} u + Q(x,t), \\ u(x,0) =& f(x) \\ u(0,t) =& A(t), u(L,t) = B(t) \end{aligned}$$

1> Homogenization:

- Equilibrium solu? Q(x,t)
- May NOT be able to reduce both PDE and BC to homo.
  choose one: PDE or BC?

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$\Rightarrow$ reference solution $r(x,t)$ s.t.

$$r(0,t) = A(t); r(L,t) = B(t)$$

- any r ***
- $r(x,t) = A(t) + \frac{x}{L}[B(t) - A(t)]$.

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2> Displacement solution

$u = v + r: \quad v(x,t) = u(x,t) - r(x,t)$

$$\partial_t v = \kappa \partial_{xx} v + \overline{Q}(x,t),$$
$$v(x,0) = f(x) - r(x,0)$$
$$v(0,t) = 0, v(L,t) = 0$$

- $\overline{Q} =?$: ***
- can we use separation of variables?
  $v(x,t) = h(t)\phi(x)$
  $\overline{Q}$ and no POS.
- Fourier series
  $v(x,t)$ " $=$ " $\sum_{n=0}^{\infty} a_n(t) \sin \frac{n\pi}{L} x$
  $\overline{Q}(x,t)$ " $=$ " $\sum_{n=0}^{\infty} q_n(t) \sin \frac{n\pi}{L} x$
  $\rightarrow$ method of eigenfunction expansion $\downarrow$

Outline

Section 8.3: Methods of eigenfunction expansion

separation of variables: homo PDE + homo BC

generalize $\rightarrow$ non-homo PDE + homogeneous BC
Seek solution of the form

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos \frac{n\pi}{L} x + b_n(t) \sin \frac{n\pi}{L} x,$$

- homo BC determines the eigenfunctions to use (sine/cosine/both, denote by $\phi_n(x)$)
- works for equation with source $\partial_t u = \kappa \partial_{xx} u + Q(x,t)$
- solve $a_n(t), b_n(t)$ from the PDE + IC (use TBTD, under conditions)

Method of eigenfunction expansion

$$\partial_t v = \kappa \partial_{xx} v + \overline{Q}(x,t),$$
$$v(x,0) = g(x)$$
$$v(0,t) = 0, v(L,t) = 0$$

$$\phi_n(x) = \sin \tfrac{n\pi}{L} x$$

$$v(x,t) \text{“}=\text{”} \sum_{n=0}^{\infty} b_n(t)\phi_n(x), \quad (b_n \text{ TBD} \downarrow)$$

$$g(x) \text{“}=\text{”} \sum_{n=0}^{\infty} b_n(0)\phi_n(x), \quad b_n(0) = ?$$

$$\overline{Q}(x,t) \text{“}=\text{”} \sum_{n=0}^{\infty} \overline{q_n}(t)\phi_n(x)$$

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$$\overline{Q}(x,t) \text{``}=\text{''} \sum_{n=0}^{\infty} \overline{q_n}(t)\phi_n(x)$$

TBTD $\partial_t v, \partial_{xx} v$ PS; $v, \partial_x v$ continuous; (BC?)$\Rightarrow$

- $$\partial_t v = \sum_{n=0}^{\infty} b_n'(t)\phi_n(x)$$
  $$\kappa\partial_{xx} v + \overline{Q}(x,t) = \sum_{n=0}^{\infty}[-\lambda_n\kappa b_n(t) + \overline{q_n}(t)]\phi_n(x)$$

  $$\Rightarrow b_n' + \lambda_n\kappa b_n(t) = \overline{q_n}(t), \quad \forall n \geq 1$$

  $$b_n(t) = b_n(0)e^{-\kappa\lambda_n t} + \int_0^t e^{-\kappa\lambda_n(t-s)}\overline{q_n}(s)ds$$

- Check: if $\overline{Q}(x,t) = 0$: $b_n(t) = b_n(0)e^{-\kappa\lambda_n t}$.

Example
Find a solution of

$$\partial_t u = \kappa \partial_{xx} u + e^{-t} \sin 3x$$
$$u(x,0) = f(x)$$
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$L = \pi$, $\lambda_n = n^2$;

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- $\overline{Q}(x,t) = e^{-t} \sin 3x + 0$
- BC$\Rightarrow$ $\phi_n(x) = \sin nx$

Seek $v(x,t) = \sum_{n=0}^{\infty} b_n(t) \phi_n(x)$

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TBTD $\partial_t v, \partial_{xx} v$ PS; $v, \partial_x v$ continuous; $\Rightarrow$

- $b_n(t) = b_n(0)e^{-\kappa\lambda_n t} + \int_0^t e^{-\kappa\lambda_n(t-s)}\overline{q_n}(s)ds$

$u(x,t) = v(x,t) + \frac{x}{\pi}$. What is u when $f(x) = sin(x)$?

Q: did we require the PDE to hold for each $x \in (0,L)$?

Outline

Method of eigenfunction expansion: seek a solution of the form

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos \frac{n\pi}{L} x + b_n(t) \sin \frac{n\pi}{L} x,$$

Q: can we use it for PDE non-homo BC directly?
e.g., seek a Fourier since series solution for

$$\begin{aligned} \partial_t u =& \ \kappa \partial_{xx} u + e^{-t} \sin 3x, \quad x \in (0,\pi) \\ u(x,0) =& f(x) \\ u(0,t) =& 0, u(\pi,t) = 1 \end{aligned}$$

When using MEE, what if

- TBTD conditions not satisfied
- More general equations $\partial_{xx} \to \frac{d}{dx}(p(x)\frac{d}{dx})$

Back to then, after chapter 5.

Outline

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