MATH 405 FALL HOMEWORK SOLUTIONS 1-5

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Note that there may be typos and errors in these solutions. Consider it an exercise to find them!

Homework 1

Chapter 0 Problem 2

Let $P(k)$ be a statement that is true or false for each natural number k. The principle of induction is as follows:

(1) Suppose $P(1)$ is true and $P(k)$ being true implies $P(k+1)$ is true for all $k \in \mathbb{N}$. Then $P(k)$ is true for all k.

The principle of strong induction is as follows.

(1) Suppose $P(1)$ is true and $P(1), ..., P(k)$ being true implies $P(k+1)$ is true for all $k \in \mathbb{N}$. Then $P(k)$ is true for all k.

To show the two are equivalent, we must show that anything proven true for all k by one can be proven true for all k by the other. Note that, trivially, proving something is true by induction implies one can prove it using strong induction. This is because strong induction assumes more than induction, and assuming more things are true will not change the result if it is proven without those additional assumptions.

Therefore, we must show that anything proven with strong induction can be proven with induction. Assume this is not the case, that is there exists a statement $P_0(k)$, indexed by the natural numbers, such that strong induction proves $P_0(k)$ is true for all k but induction does not. Note: the difficulty of the problem is the difference between something being true and something being proven to be true by a specific method.

By assumption, there exists a smallest k' such that induction cannot prove $P(k')$ but strong induction can. In this case we have

$$P(k'-1) \text{ is true but it does not imply } P(k') \text{ is true.}$$

However, induction allows us to assume $P(1)$ is true. And, as k' is the smallest such that the implication fails, we know $P(2)$ is true. We also know $P(3)$ is true, and so on, up to

$P(k'-1)$. However, $P(k')$ can be proven by *strong* induction. We have show the requirements for strong induction are met, and so $P(k')$ must be true. However, we only used the assumptions of regular induction, so regular induction can prove $P(k'-1)$ implies $P(k')$. This is a contradiction, and therefore k' cannot exists, and so the two methods must be equivalent.

Chapter 0 Problem 7

(1) This is a Venn diagram with the middle not shaded in but everything else is.
(2) Let $x \in A\Delta B$. Suppose without loss of generality that $x \in A$ (otherwise, rename A and B). Then $x \notin B$, and so $x \in A \setminus B$. So $x \in (A \setminus B) \cap (B \setminus A)$ and therefore, as x was arbitrary, $A\Delta B \subset (A \setminus B) \cap (B \setminus A)$.
Now suppose $x \in (A\setminus B)\cap(B\setminus A)$. Then either $x \in A\setminus B$ or $x \in B \setminus A$. Assume without loss of generality that $x \in A \setminus B$. Then $x \in A$ but $x \notin B$, and so $x \in A\Delta B$. Therefore, as x was arbitrary, $(A\setminus B)\cap(B\setminus A) \subset A\Delta B$. As each is a subset of the other, the two sets must be equal.
(3) Let $x \in A\Delta B$. Then $x \in A \cap B$ as x must be in either A or B. However, $x \notin A \cup B$ by definition, and so $x \in (A \cup B)(A \cap B)$ and so as x was arbitrary we have $A\Delta B \subset (A \cup B) \setminus (A \cap B)$.
Suppose $x \in (A \cup B) \setminus (A \cap B)$. Then $x \in A$ or $x \in B$. Assume A without loss of generality. Then $x \notin B$ as otherwise it would be in the intersection. As such $x \in A\Delta B$ as it is in one but not the other. As x was arbitrary, we have $(A \cup B) \setminus (A \cap B) \subset A\Delta B$.
As each is a subset of the other, the two sets must be equal.

Chapter 0 Problem 10

(1) Suppose f is not injective. Then there exits a_1, a_2 such that $f(a_1) = f(a_2)$. But this forces $(g \cdot f)(a_1) = (g \cdot f)(a_2)$, contradicting the injectivity of $g \cdot f$.
(2) Suppose g is not surjective. Then there exists $c \in C$ such that c is not in the range of g. But the range of $g \cdot f$ is a subset of the range of g, and so $g \cdot f$ is not surjective.
(3) Let $f : \{0\} \to \mathbb{R}$ be given by $f(0) = 1$. Let $g : \mathbb{R} \to \{2\}$ be given by $g(x) = 2$ for all $x \in \mathbb{R}$. Then $g \cdot f : \{0\} \to \{2\}$ bijectively, but clearly neither f nor g are bijective.

Chapter 0 Problem 12

The largest subset of A has cardinality n. Enumerate the elements of A as $a_1, a_2, ..., a_n$. We also consider the set of n-tuples with entries 0 or 1. That is, all sets of the form $(1, 1, 1, ...)$, $(0, 1, 1, ...)$, $(1, 0, 0, ...)$ and so on. The number of these sets is 2^n has each slot has two choices: either containing a 0 or containing a 1, and there are n slots.

We claim the collection of these n-tuples is in bijection with the power set of A. We define a map between the two as follows: If P_S is an element of the power set of A, it contains some number of $a_1, ..., a_n$. The map f sends this set to the n-tuple generated by having 1 in the first slot if P_S contains a_1 and 0 otherwise, a 1 in the second slot if P_S contains a_2 and a 0 otherwise, and so on. Showing this map is a bijection follows immediately from the definition as every n-tuple is sent uniquely to a subset and every subset is sent uniquely to an n-tuple. Therefore the power set has cardinality 2^n. Note the empty set gets sent to the n-tuple with all 0s.

Hint: If the above explanation is still mysterious, write out the sets and maps in the case of $n = 2, 3$.

Chapter 0 Problem 14

We proceed by induction. The claim is obviously true for $n = 1$. Next, assume

$$1^3 + 2^3 + ... + (n-1)^3 = \left(\frac{(n-1)n}{2}\right)^2.$$

We then have

$$\left(\frac{(n-1)n}{2}\right)^2 + n^3 = \frac{(n^2 - 2n + 1)n^2}{4} + n^3 = \frac{n^4 + 2n^3 + n^2}{4} = \left(\frac{(n+1)n}{2}\right)^2.$$

And the induction closes.

Chapter 0 Problem 20

Consider $A_k = \{0\} \cup [k, \infty)$

Chapter 0 Problem 24

(1) Let $f : \mathbb{N} \to B$ be a bijection between the natural numbers and the set B, which is guaranteed to exist by the cardinality of B. This allows us to order the elements of B in the following way. To each $b \in B$, assign a natural number given by $f^{-1}(B)$. We say that $b < b'$ if $f^{-1}(b) < f^{-1}(b')$, with similar notation for greater than,

equal to, and so on. Note this is well defined precisely because f is a bijection.
Assume A is non-empty. If it is empty then we are done. We define a function g in the following way:
(a) $g(1)$ is the smallest element of A.
(b) $g(2)$ is the smallest element of $A \setminus \{g(1)\}$
(c) $g(3)$ is the smallest element of $A \setminus \{g(1), g(2)\}$
(d) And so on

If at any point the set $\{g(1), ..., g(k)\} = A$ for a finite k, we do not define g any further. If A is finite then $g : \{1, ..., k\} \to A$ in a bijective manner. If A is not finite then $g : \mathbb{N} \to A$ in a bijective manner. In any case, A is countable.

(2) The above construction shows that either A is empty, finite, or in bijection with the natural numbers. If $|A| < |\mathbb{N}|$, the process defined before must terminate, and A must be empty or finite.

Chapter 0 Problem 25

Let A be the countably infinite subset of S. If S has a cardinality larger than A, then $S \setminus A$ must have the same cardinality as A (so there exists a bijection between the two). If not, then S can be written as the union of finitely many countable sets and would be countable.

Now assume S is countably infinite. Then, by definition, there exists a bijection $f : \mathbb{N} \to S$. Let

$$A = \{s \in S : f(n) = s, \text{ and } n \text{ is an even number}\}.$$

Then, f is a bijection, the set $S \setminus A$ is countably infinite as it is in bijection with the set of odd natural numbers. As $S \setminus A$ has the same cardinality as S there exists a bijection between them, so we are done.

Homework 2

Chapter 1.1 Problem 2

We induct on the cardinality of A. If $|A| = 1$, then $A = \{a\}$ for some $a \in S$. A is then bounded as every element of A is $\leq a$ and $\geq a$. This also shows A has an infimum and supremum as a is both. It is easily checked a has the required property.

For the inductive step, we will only show that A contains its supremum and infimum. It immediately follows A is bounded. We focus first on the supremum. Assume any set with cardinality n has and contains its supremum. Assume $|A| = n+1$. We write $A = \{a_1, ..., a_n\}\{a_{n+1}\}$ where a_{n+1} is any element of the set. We can always label A in this way because it is finite.

The set $\{a_1, ..., a_n\}$ has and contains its supremum by the inductive hypothesis. Call that element a^*. There are two cases. If $a^* < a_{n+1}$, then a_{n+1} is easily seen to be a supremum for A. If not, then a^* is the supremum of A. In either case A contains its supremum and we are done.

The case for infimum follows identically, except with different conclusions based on the comparison of the infimum from the inductive step and the elements a_{n+1}.

Chapter 1.1 Problem 8

(1) The value of $0+0, 0+1$ and $0+2$ is fixed by the definition of 0. The same is true for multiplication by 1. We must also have $0x = 0$ as

$$0x = 0x + 0 \iff 0 = 0 + 0x^{-1} \implies 0 = 0x^{-1}.$$

This is true for every $x^{-1} \in F$, we have that multiplication by 0 is always 0 when $x \neq 0$. In the case where $x = 0$, we have $0 = (1+0)0 \implies 0 \cdot 0 = 0$. This only leaves $2 \cdot 2$. Suppose $2 \cdot 2 = 2$. Then by applying the multiplicative inverse we have $2 = 1$ which is false. Suppose $2 \cdot 2 = 0$. Then by applying the multiplicative inverse we have $2 = 0$ which is false. Therefore $2 \cdot 2 = 1$. This concludes the multiplication part of the problem. For addition we must fill in 1+1,1+2, and 2+2. Suppose $1+2 = 1$. Then $2 = 0$ which is impossible. Additionally, suppose $1+2 = 2$. Then $1 = 0$ which is impossible. So $1+2 = 0$. Suppose $1+1 = 0$. Then $1+1 = 1+2$ which implies $1 = 2$, which is impossible. Suppose $1+1 = 1$. Then $1 = 0$ which is impossible. Suppose $2+2 = 0$. Then $1+1 = 0$ by applying the multiplicative inverse which is impossible. Suppose $2+2 = 2$. Then $2 = 0$ which is impossible. Therefore $2+2 = 1$. This concludes the addition part of the problem.

(2) Assume F is an ordered field. Because $2^{-1} = 2$ and $1^{-1} = 1$, we cannot order 1 and 2 as not matter what we choose, property (v) will be violated.

Chapter 1.1 Problem 12

We claim the set $\{n\cdot 1 : n \in \mathbb{N}\}$ is countably infinite. Here, we write $n\cdot 1 = 1+...+1$ where n copies of 1 are added together. Suppose this set is finite. Then there must be some m such that $m \cdot 1 = 0$. However, in any ordered field we have $1 > 0$ as 1 is a square. The existence of such an m violates property *vii* as $1 = 1+0 < 1+1 = 2$. Repeatedly adding 0 on the left and 1 on the right reaches a contradiction as must eventually have $1 < m \cdot 1 = 0$.

Chapter 1.1 Problem 13

The only difference between $\mathbb{N}$ and $\mathbb{N}_\infty$ is the element ∞, and so to show $\mathbb{N}_\infty$ is an ordered set we only need to show $k < \infty$ for all $k \in \mathbb{N}$ which we assume (also nothing the obvious $\infty = \infty$).

Let $E \subset \mathbb{N}\infty$. Suppose E is bounded. Then it is finite and has an contains its supremum by problem 2. Suppose it is not bounded but does not contain ∞. Then ∞ is the least upper bound. By assumption it is an upper bound. It must be the smallest as any smaller number k_0 is in $\mathbb{N}$, but we assumed E was not bounded and so k_0 cannot be an upper bound. Now suppose $\infty \in E$. Then ∞ is obviously the supremum as it is greater than or equal to every element of $\mathbb{N}_\infty$.

Chapter 1.2 Problem 9

We prove the result for supremums first. To show $\sup A + \sup B$ is an upper bound for C, one simply uses the ordered field property that $x \leq a$ and $y \leq b$ implies $x + y \leq a + b$. Apply this with $a \leq \sup A$ and $b \leq \sup B$. We must now show it is the least upper bound. Suppose there exists a smaller upper bound called U. Then $U < \sup A + \sup B$ - and so $U - \sup B < \sup A$. We will show that $U - \sup B$ is an upper bound for for A. Let $a \in A$. Then $a + b \in C$ for every $b \in B$. So $a + b \leq U$. And so $a \leq U - b$. As this must hold for *every* $b \in B$, there must exit a sequence $b_n \to \sup B$ (otherwise $\sup B$ would not be the supremum). We must then have, for every $a \in A$ that $a \leq U - b_n$. By taking the limit in n of both sides we have $a \leq U - \sup B$ which implies $U - \sup B$ is an upper bound smaller than $\sup A$, which is a contradiction.

Therefore, $\sup C = \sup A + \sup B$. The proof for infimums can be proven the following way. For any bounded subset of real numbers A, define $-A = \{-a : a \in A\}$. It is easy to show that $\inf(-A) = -\sup A$ and vice versa. The proof above then goes through by considering $-A, -B$ and $-C = -A - B = -(A + B)$ all defined appropriately.

Chapter 1.1 Problem 10

This follows the same outline as the previous problem. It is clear $\sup C$ is an upper bound. To show it is a least upper bound, assume there exists U such $ab \le U$ for every $a \in A$ and $b \in B$ and $U < (\sup A)(\sup B)$. Assume $\sup B \ne 0$. If $\sup B = 0$ then $B = \{0\}$ and the claim is trivial. Then $a \le \frac{U}{b}$ for every $b \ne 0$. If the supremum is attained, one can just conclude that $U/\sup B$ is an upper bound for A. If not take a sequence $b_n \to \sup B$. Applying limits gives that $U/\sup B$ is an upper bound of A less than $\sup A$.

One can also deduce the result for infimums by either repeating the above proof or by the following observations. If either $\inf A$ or $\inf B$ is 0, then $\inf C = 0$ and the claim is trivial. As such, assume A and B are bounded from below by positive numbers. Define A^{-1} and B^{-1} in the obvious way and prove a relation between $\sup A^{-1}$ and $\inf A$. The result follows.

Chapter 1.2 Problem 12

Suppose not. then there exists $\epsilon_0 > 0$ such that $S \cap (\sup S - \epsilon_0, \sup S) = \emptyset$. But then $\sup S - \frac{\epsilon_0}{2}$ is a least upper bound which is a contradiction of the definition of $\sup S$.

Chapter 1.2 Problem 13

We induct. The claim for $n = 1$ is trivial. Assume $1 + nx \le (1+x)^n$. Then we have

$$1 + (n+1)x = 1 + nx + x \le (1+x)^n + x.$$

We also have $(1+x)^{n+1} = (1+x)^n(1+x) = (1+x)^n + x(1+x)^n$.

So we would be done if we prove $x \le x(1+x)^n$. This is clearly true if $0 = x$ or $0 < x$. Suppose $-1 < x < 0$ (the lower bound comes from the assumption of the problem). Then we want to show $1 \ge (1+x)^n$ after dividing through by x. However the bound on x implies $0 < 1 + x < 1$. One can prove $1 \ge (1+x)^n$ by proving that the product of any two positive numbers smaller than 1 is smaller than 1. this is easily done by applying property v and ii of ordered fields. And so we are done.

Homework 3

Chapter 1.3 Problem 2

We will handle part a and b simultaneously. Suppose $x = y$. Then $\frac{1}{2}(x + y \pm |x - y|) = \frac{1}{2}(x + y) = x = y$. And so in either case we have the conclusion as $\max\{x, y\} = x = y = \min\{x, y\}$.

Next, assume $x - y > 0$. Then the max is x and the min is y. We then compute

$$|x - y| = x - y.$$

The result immediately follows using $\max\{x, y\} = x$ and $\min\{x, y\} = y$. The case for $x - y < 0$ is similar with $|x - y| = y - x$ and $\max\{x, y\} = y$ and $\min\{x, y\} = x$.

Chapter 1.3 Problem 7

(1) As $\sup(-f) = -\inf(f)$, proving the result for supremums immediately implies it for infimums. We focus on the case of supremums. Note that we have

$$f(x) + g(x) \leq \sup_{x\in D} f(x) + \sup_{x\in D} g(x)$$

just by the definition of supremum. Note the right hand side is *not* a function of x. As such we can apply $\sup_{x\in D}$ to both side. As the right hand side is constant we have

$$\sup_{x\in D}\Big(f(x) + g(x)\Big) \leq \sup_{x\in D}\Big(\sup_{x\in D} f(x) + \sup_{x\in D} g(x)\Big) = \sup_{x\in D} f(x) + \sup_{x\in D} g(x),$$

which gives the conclusion.

(2) For strict equalities, consider $f(x) = g(x) = 0$. Any functions f and g which have the same sign will work here.

Chapter 1.3 Problem 8

(1) Suppose $|f(x)| \leq M$. Then $|\alpha f(x)| \leq \alpha|f(x)| \leq M$. So αf is bounded.

(2) Suppose $|f(x)| \leq M$ and $|g(x)| \leq N$. Then $|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M + N$. So $f + g$ is bounded.

Chapter 1.3 Problem 9(ab)

(1) Write $f = f + g - g$. By the triangle inequality $|f| \leq |f - g| + |g|$. As $f = g$ and g are bounded, so is f.

(2) Let $f(x) = x$ and $g(x) = -x$ with $D = \mathbb{R}$.

Chapter 1.4 Problem 1

Note $b \neq 0$ as otherwise the set is empty. Let $f(x) = \frac{1}{b}(x-a)$. Then $f : (a,b](0,1)$ is a bijection. Indeed, $f(x) = f(y)$. Then $\frac{1}{b}(x-a) = \frac{1}{b}(y-a) \iff x = y$ by arithmetic. Now let $z \in (0,1]$. We claim $f(bz+a) = a$. Indeed, $f(ba+z) = \frac{1}{b}(bz+a-a) = z$. Moreover, $bz+a \in (a,b]$ if $z \in (0,1]$.

Chapter 1.4 Problem 2

We know $\tan(x)$ is bijection from $(-\frac{\pi}{2}, \frac{\pi}{2})$. Let $g(x)$ be a bijection from $(-1,1)$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$ which is given by $g(x) = \frac{pi}{2}x$. Then $\tan \cdot g \cdot f$ is a bijection from $[-1,1]$ to $\mathbb{R}$ as the composition of bijective functions is bijective.

Chapter 1.4 Problem 6

(1) Consider $I_n = (a - \frac{1}{n}, a + \frac{1}{n})$. Clearly $[a,b] \subset \cap_n I_n$. By the Archimedean property, every element of the intersection is $\leq b$ and $\geq a$, so $\cap_n I_n = [a,b]$.
(2) Let $I_n = [a + \frac{1}{n}, b - \frac{1}{n}]$ where we only consider n large enough so that I_n is non-empty. Then $(a,b) = \cup I_n$. For any $x \in (a,b)$ there exists an n such that $a + \frac{1}{n} < x < b - \frac{1}{n}$. Moreover, a, b are not in the set for every n. Same holds for any number larger than b or smaller than a trivially.
(3) We may think of $[a_\lambda, b_\lambda]$ as the intersection of $\{x \in \mathbb{R} : x \geq a_\lambda\}$ and $\{x : x \leq b_\lambda\}$. The intersection of every open interval is then

$$\{x \in \mathbb{R} : x \leq b_\lambda \text{ and } x \geq a_\lambda \, \forall \lambda\}.$$

Let $B = \inf_\lambda b_\lambda$. Let $A = \sup_\lambda a_\lambda$. If A or B are infinite, then the intersection is empty. If $A = B$, then the intersection is the singleton $\{A\}$. Suppose the two are not equal and are both finite. Then we claim the intersection is $[A, B]$. This follows from the definition of infimum and supremum. All points larger than a_λ for every λ are given by $[A, \infty$. All points smaller than b_λ for each λ are given by $(-\infty, B]$. Computing the intersection is obvious.

Chapter 1.4 Problem 7

For each $s \in S$ define a function f where $f(x) = q_s$ where q_s is a rational number contained in $s = (a,b)$. The existence of such a q is guaranteed by density of the rationals. One can easily show this function is a bijection from S to a subset of $\mathbb{Q}$. Surjectivity follows by considering $Im(f) \subset \mathbb{Q}$. Injectivity follows from the disjoint property of S.

Homework 4

Chapter 2.1 Problem 7

(1) Let $\epsilon > 0$. Let N_ϵ be such that for every $n \geq N_\epsilon$ we have $|x_n - 0| < \epsilon$. Then clearly the same holds true for $||x_n| - 0|$. The reverse direction also holds using the fact that $||x|| = |x|$.
(2) The sequence $x_n = (-1)^n$ works.

Chapter 2.1 Problem 12

Bounded implies the supremum exists. For every $\epsilon > 0$, we know there exists $x \in S$ such that $\sup S - \epsilon < x \leq \sup S$. Let x_n be such that the previous statement holds for $\epsilon = \frac{1}{n}$. The conclusion follows. A similar proof holds for the infimum or from the supremum result by considering the negative of the set..

Chapter 2.1 Problem 22

Denote x_{even} the limit of $\{_{2n}\}$, denote x_{odd} the limit of $\{x_{2n+1}\}$, and x the limit of $x_{\{}3n\}$.

Consider the subsequence of $x_{\{}3n\}$ given by the even indices, so $n = 2k$ for some k. The subsequence is convergent as the original subsequence is convergent and converges to x. However, it is also a subsequence of $\{_{2n}\}$ and therefore must also converge to x_{even}. Therefore $x = s_{even}$. By consider $n = 2k + 1$ we can get a subsequence of $x_{\{}3n\}$ that is also a subsequence of $\{x_{2n+1}\}$. This similarly implies $x_{odd} = x$. Therefore $x_{odd} = x_{even.}$

Fix $\epsilon > 0$. Let N_{even} be a number such that $|x_{2n} - x| < \epsilon$ for $n > N_{even}$. Use a similar definition for N_{odd}. Let $N = \max\{N_{odd}, N_{even}\}$. Then for every $\epsilon > 0$, $n > N$ implies $|x_n - x| < \epsilon$ so we have convergence.

Chapter 2.1 Problem 23

Let x be the limit of the subsequence. Then x is an upper bound for x_n. Suppose there exists an $x_k > x$. But as x_n monotone then $x > x_{n_k} > x_k$ which is a contradiction. Therefore x_n converges as it is bounded and monotone. The result follows.

Chapter 2.2 Problem 9

Define $a_n = x_n - x$ which is a well defined sequence as $x \in \mathbb{R}$. Apply the ratio test for sequences to a_n to get $\lim_{n\to\infty} a_n = 0$. the conclusion follows.

Chapter 2.2 Problem 11

This follows Example 2.2.8. We write

$$x_{n+1} = \frac{2x_n^2 - x_n^2 + r}{2x_n} = \frac{x_n^2 + r}{2x_n}.$$

If $x_1 > 0$ this is always positive. If $x_1 < 0$ then this is always negative by induction.

We now prove monotonicity. Consider

$$\frac{x_n^2 + r}{2x_n} - x_n = \frac{x_n^2 - 2x_n^2 + r}{2x_n} = \frac{r - x_n^2}{2x_n}.$$

If $x_n > 0$ and $r > x_n^2$ this is decreasing. If $0 < r < x_n^2$ this is increasing. If $x_n < 0$ and $r > x_n^2$ the sequence is decreasing. If $x_n < 0$ and $r < x_n^2$ the sequence is increasing. One can use induction to prove that in each case it is monotone.

Taking the limits of both sides (using tails to make this formal) we get

$$x = x - \frac{x^2 - r}{x} \implies x = \pm\sqrt{r}.$$

Using monotonicity one can conclude $x = \sqrt{r}$ when $x_1 > 0$ and $x = -\sqrt{r}$ when $x_1 < 0$.

Chapter 2.2 Problem 14

We have $x_{n+1} - x_n = x_n^2$ so the sequence is monotone increasing. This immediately implies the sequence diverges if $x_1 > 0$ as $x > y \implies x^2 > y^2$ when $x > y > 0$, so the amount being added to each term in the sequence increases.

Suppose $x_1 < -1$. Then $x_2 = x_1^2 + x_1 > 0$, and so we may apply the previous argument to get divergence. Clearly the sequence converges if any element equals 0, as any after must be 0. This implies the sequence converges for $x_1 = 0, -1$. Suppose $-1 < x_1 < 0$. Then $|x_1|^2 < |x_1|$. This implies $x_1^2 + x_1$ is larger than x_1 but still negative. By induction x_n is a monotone increasing sequence bounded above 0. It therefore converges. Taking limits we immediately conclude

$$x = x^2 + x \implies x = 0.$$

This handles all cases.

Chapter 2.2 Problem 15

Repeat the proof of Example 2.2.14 except consider the sequence $\frac{n^2+1}{(1+\epsilon)^n}$ Crucially $\lim_{n\to\infty} \frac{(n+1)^2+1}{n^2+1} = 1$ and the conclusion follows.

Homework 5

Chapter 2.3 Problem 5

(1) Note $|x_n| = \frac{1}{n}$. We have $\lim_{n\to\infty} |x_n| = 0$. Therefore $\lim_{n\to\infty} x_n = 0$. So the lim sup and the lim inf must also be 0.
(2) Write $x_n = (-1)^n - (-1)^n/n$. The second term converges to 0, so we can analyze just the first. This sequence is bounded by 1. It also has a subsequence that goes to -1 and 1, so 1 is the limsup and -1 is the liminf.

Chapter 2.3 Problem 9

Once we have a sequence of distinct numbers we are done. By BW we know this bounded sequence has a convergent subsequence. The limit of this subsequence is a cluster point. As each element on the list is unique, we can apply the definition of convergence to get an element of x_n as close to the limit point as we like that is not x.

Chapter 2.3 Problem 11

(1) If not, one can construct an unbounded subsequence which contradicts the assumption.
(2) If x_n does not converge to x the limsup and liminf must be distinct and so one must not be x. As we have proven boundedness these must be finite. Then construct a subsequence that approaches the limsup or the liminf (whichever is not x). This cannot have a subsequence that approaches x.

Chapter 2.3 Problem 16

As a_l is the sup of a set but not a member, there is a subsequence approaching it. Additionally, a_j is a strictly decreasing sequence in terms of j. However, the tail of the subsequence that approaches a_l is contained within $\{x_n : n > j\}$ for every j, and so $a_j \leq a_l$. This gives the result.

Chapter 2.3 Problem 18

Suppose not. Then there exists a subsequence going to positive or negative infinity. Therefore at least one is not bounded.

Chapter 2.4 Problem 2

Consider $|x_n - x_k|$. By repeated use of the triangle inequality we can write this as

$$\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + ... + |x_{k-1} - x_k|.$$

We can then write use the bound given repeatedly to get

$$\leq |x_n - x_{n+1}|(1 + C + ... + C^{n-k+1}).$$

Next we may apply the formula repeatdly again to get

$$\leq |x_1 - x_2|C^{n-1}(\frac{1 - C^{n-k}}{1 - C}).$$

As $|x_1 - x_2|$ is fixed and $\frac{1-C^{n-k}}{1-C}$ is bounded above by $\frac{1}{1-C}$, we may make this as small as we like by taking n to be large. A quick rewriting of this statements shows that the sequence is Cauchy.

Chapter 2.4 Problem 4

For every $\epsilon > 0$ select k_ϵ such that $y_{k_\epsilon} < \epsilon$. This works in the definition of Cauchy.

Chapter 2.4 Problem 5

For every $\epsilon > 0$ there exists a N_ϵ such that $|x_n - x_k| < \epsilon$ when $n \geq N_\epsilon$. We also have two elements, one of which is positive and one of which is negative. This implies every element with $j > N_\epsilon$ must be within ϵ of 0. The result follows.