Real Analysis I: Midterm Exam Sample, 10/8/2025

Your name: ________________________

This is a closed-book in-class exam. **Cell phones are NOT allowed in the exam.**

1. (10 pts) True or False:

(1)_____: The set $\mathbb{Q}$ has the least-upper-bound property.

(2)_____: Let $f : D \to \mathbb{R}$ be a bounded function. Then, there exists $x \in D$ such that $f(x) = \sup_{y \in D} f(y)$.

(3)_____: Let $f, g : D \to \mathbb{R}$ be bounded functions and $f(x) \leq g(x)$ for all $x \in D$. Then, $\sup_{x \in D} f(x) \leq \inf_{x \in D} g(x)$.

(4)_____: A divergent sequence can have a convergent subsequence.

(5)_____: If both subsequences $\{x_{2n}\}_{n \geq 1}$ and $\{x_{2n-1}\}_{n \geq 1}$ converge, then $\{x_n\}$ itself converges.

Answer. FFFTF

2. (20 = 5+15 pts) Supremum and infimum.
(a) State the definition of the supremum of a set $A \subset \mathbb{R}$.
(b) Let $A, B \subset \mathbb{R}$ be nonempty bounded sets. Let $C = \{a + b : a \in A, b \in B\}$. Prove that

$$\sup C = \sup A + \sup B.$$

Answer (a). A real number s is called the supremum (or least upper bound) of a set $A \subset \mathbb{R}$ if it satisfies the following two conditions: (1) s is an upper bound of A, i.e., $a \leq s$ for all $a \in A$. (2) s is the least such upper bound, i.e., if t is any upper bound of A, then $s \leq t$.
Answer (b). Let $s = \sup A$ and $t = \sup B$. We show that $\sup C = s + t$. Since $a \leq s$ for all $a \in A$ and $b \leq t$ for all $b \in B$, we have $a + b \leq s + t$ for all $a \in A$ and $b \in B$. Thus, $s + t$ is an upper bound of C. Now, suppose there exists another upper bound u of C such that $u < s + t$. Then, we can choose $\epsilon = (s + t - u)/2 > 0$. By the definition of supremum, there exist elements $a_0 \in A$ and $b_0 \in B$ such that $s - \epsilon < a_0 \leq s$ and $t - \epsilon < b_0 \leq t$. Adding these inequalities, we get $s + t - 2\epsilon < a_0 + b_0 \leq s + t$. This implies $u < a_0 + b_0$, which contradicts the fact that u is an upper bound of C. Therefore, no such u exists, and hence $s + t$ is the least upper bound of C. Thus, we conclude that $\sup C = s + t$.

3. (20 points) Limit superior and limit inferior.
(a) State the definition of $\liminf_{n\to\infty} x_n$ for a sequence $\{x_n\}_{n=1}^{\infty}$.
(b) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that both limit superior and limit inferior are finite. Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded.

Answer (a). The limit inferior is defined as

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right).$$

This means that for each n, we consider the infimum of the tail of the sequence starting from index n, and then take the limit of these infima as n approaches infinity (the limit exists since the sequence is monotonic increasing).
Answer (b). Let $L = \limsup_{n\to\infty} x_n$ and $l = \liminf_{n\to\infty} x_n$. Assume, for the sake of contradiction, that the sequence is not bounded above, that is for any M, there exists an integer $n > M$ such that $x_n > L+1$. Thus, for $M = 1$, there exists an integer $n_1 > 1$ such that $x_{n_1} > L+1$; for $M = n_{k-1} > k-1$ with $k \geq 1$, there exists an integer $n_k > \max\{n_{k-1}, k\}$ such that $x_{n_k} > L+1$. Then, $y_k := \sup_{n\geq k} x_n \geq x_{n_k} > L+1$ and $\limsup_{n\to\infty} x_n = \lim_k y_k > L+1$. This contradicts the assumption that $\limsup_{n\to\infty} x_n = L$. Therefore, the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded above.

Similarly, assume that the sequence is not bounded below, that is for any N, there exists an integer $m > N$ such that $x_m < l-1$. Thus, for $N = 1$, there exists an integer $m_1 > 1$ such that $x_{m_1} < l-1$; for $N = m_{j-1} > j-1$ with $j \geq 1$, there exists an integer $m_j > \max\{m_{j-1}, j\}$ such that $x_{m_j} < l-1$. Then, $z_j := \inf_{n\geq j} x_n \leq x_{m_j} < l-1$ and $\liminf_{n\to\infty} x_n = \lim_j z_j < l-1$. This contradicts the assumption that $\liminf_{n\to\infty} x_n = l$. Therefore, the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded below.

Therefore, the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded.

4. (25 pts) Cauchy sequence.
(a) State the definition of a Cauchy sequence in $\mathbb{R}$.
(b) Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}$. Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded.

Answer (a). A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is Cauchy if for every $\epsilon > 0$, there exists a positive integer N such that for all integers $m, n > N$, the distance between the terms x_m and x_n is less than ϵ, i.e.,

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall m, n > N,\ |x_m - x_n| < \epsilon.$$

Answer (b). Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, for $\epsilon = 1$, there exists a positive integer N such that for all integers $m, n > N$, we have $|x_m - x_n| < 1$. In particular, for any $n > N$, we have $|x_n - x_{N+1}| < 1$. This implies that x_n lies within the interval $(x_{N+1} - 1, x_{N+1} + 1)$ for all $n > N$. Therefore, the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded.

5. (25 pts) Series.
(a) Prove that $\lim_{n\to\infty} n^{1/n} = 1$. (Hint: use ratio test on $\{\frac{n}{(1+\epsilon)^n}\}$ to show $n < (1+\epsilon)^n$ for large n.)
(b) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} n\frac{(x-2)^n}{3^n}$.

Answer (a). Let $\epsilon > 0$ be given. We need to find an integer N such that for all $n > N$, we have $|n^{1/n} - 1| < \epsilon$. This is equivalent to showing that $1 - \epsilon < n^{1/n} < 1 + \epsilon$ for all $n > N$. For $n > 1$, we have $n^{1/n} > 1 > 1 - \epsilon$. Now, we need to show that $n^{1/n} < 1 + \epsilon$ for sufficiently large n. This is equivalent to showing that $n < (1+\epsilon)^n$. Consider the sequence $\{a_n\}$ defined by $a_n = \frac{n}{(1+\epsilon)^n}$. We will use the ratio test to show that $\lim_{n\to\infty} a_n = 0$. We compute the ratio:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\frac{1}{(1+\epsilon)^{n+1}}}{n\frac{1}{(1+\epsilon)^n}} = \frac{(n+1)}{n(1+\epsilon)} = \frac{1+\frac{1}{n}}{1+\epsilon}.$$

Taking the limit as $n \to \infty$, we find $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{1+\epsilon} < 1$. Thus, by the ratio test, the series $a_n \to 0$. Therefore, there exists an integer N such that for all $n > N$, we have $a_n < 1$, which implies that $n < (1+\epsilon)^n$, equivalently, $n^{1/n} < 1 + \epsilon$. Hence, for all $n > N$, we have $|n^{1/n} - 1| < \epsilon$, i.e., $\lim_{n\to\infty} n^{1/n} = 1$.

Answer (b). Approach 1: the radius of convergence is $R = \lim_{n\to\infty}(\frac{n}{3^n})^{\frac{1}{n}} = \lim_{n\to\infty} n^{1/n}3 = 3$, since $\lim_{n\to\infty} n^{1/n} = 1$ by part (a).

Approach 2: Since the series $\sum_{n=1}^{\infty} n\frac{(x-2)^n}{3^n}$ is a power series in $(x-2)$, we can find its radius of convergence using the ratio test. Consider the general term:

$$a_n = n\frac{(x-2)^n}{3^n}.$$

We compute the ratio:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\frac{(x-2)^{n+1}}{3^{n+1}}}{n\frac{(x-2)^n}{3^n}} = \frac{(n+1)(x-2)}{3n}.$$

Taking the limit as $n \to \infty$, we find:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)|x-2|}{3n} = \frac{|x-2|}{3}.$$

By the ratio test, the series converges if this limit is less than 1:

$$\frac{|x-2|}{3} < 1 \implies |x-2| < 3,$$

and it diverges if the limit is greater than 1. Thus, the radius of convergence is $R = 3$.