Real Analysis I: Final Exam

Your name: ________________________

This is a closed-book 3-hour in-class exam. **Cell phones and other electronic devices are NOT allowed in the exam.**

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1. (20 pts) True or False. For each statement, circle **T** or **F**. No justification is required.

(1)_____: Every Cauchy sequence in $\mathbb{R}$ is bounded.

(2)_____: If a sequence $\{x_n\}$ has two distinct subsequences that converge to the same limits, then $\{x_n\}$ converges.

(3)_____: If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.

(4)_____: If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(5)_____: If $f : [0,1] \to \mathbb{R}$ is continuous and differentiable on $(0,1)$, then f has a maximum and a minimum on $[0,1]$.

(6)_____: If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in \mathbb{R}$, then f is strictly increasing on $\mathbb{R}$.

(7)_____: If $f : [0,1] \to \mathbb{R}$ is Riemann integrable on $[0,1]$, then f must be continuous on $[0,1]$.

(8)_____: If $f_n : [0,1] \to \mathbb{R}$ are continuous and $f_n \to f$ pointwise on $[0,1]$, then f is continuous.

(9)_____: If $f_n \to f$ uniformly on $[0,1]$ and each f_n is bounded, then f is bounded.

(10)_____: If $f_n \to f$ uniformly on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

Solutions (pattern). T, F, T, T, T, T, F, F, T, T.

2. (20 pts) Sequences and series.

(a) (10 pts) Define a sequence $\{x_n\}$ by $x_1 = 1$ and

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{2}{x_n}\Big), \qquad n \geq 1.$$

Show that $\{x_n\}$ converges and find its limit.

(b) (10 pts) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and compute its sum.

Solution 2(a). First note that $x_n > 0$ for all n by induction. For $x > 0$ define $g(x) = \frac{1}{2}\big(x + \frac{2}{x}\big)$. Note that

$$g(x) = \sqrt{2} + \frac{(x-\sqrt{2})^2}{2x} \geq \sqrt{2}, \forall x > 0.$$

Furthermore,

$$x - g(x) = \frac{1}{2}\Big(x - \frac{2}{x}\Big) = \frac{x^2-2}{2x}.$$

If $x > \sqrt{2}$, then $x^2 > 2$ so $x - g(x) > 0$, hence $g(x) < x$. We check that $x_1 = 1 < \sqrt{2}$ and $x_2 = g(1) = \frac{3}{2} > \sqrt{2}$, and one shows by induction that $x_n \geq \sqrt{2}$ for all $n \geq 2$. Moreover, for $n \geq 2$ we have $x_{n+1} = g\{x_n\} \leq x_n$, so $\{x_n\}_{n\geq 2}$ is decreasing and bounded below by $\sqrt{2}$. Hence $\{x_n\}$ converges; let $\lim_{n\to\infty} x_n = L$. Passing to the limit in the recursion gives

$$L = \frac{1}{2}\Big(L + \frac{2}{L}\Big),$$

so $L^2 = 2$ and $L > 0$, hence $L = \sqrt{2}$.

Solution 2(b). We use partial fractions: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Hence the Nth partial sum is

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N}\Big(\frac{1}{n} - \frac{1}{n+1}\Big) = 1 - \frac{1}{N+1}.$$

Letting $N \to \infty$ we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

3. (20 pts) Continuity and differentiability.

(a) (5 pts) State and prove the Intermediate Value Theorem for continuous functions on an interval.

(b) (7 pts) Show that the equation $x^3 + x - 1 = 0$ has a unique real solution.
(c) (8 pts) Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that there exists $c \in (a, b)$ such that

$$\int_a^b f(x)\,dx = f(c)\,(b - a).$$

Solution 3(a). (Any correct statement and proof of the Intermediate Value Theorem earns full credit.)

Solution 3(b). Define $f(x) = x^3+x-1$. The function f is continuous on $\mathbb{R}$. We have $f(0) = -1 < 0$ and $f(1) = 1 > 0$, so by the IVT there exists $c \in (0, 1)$ with $f(c) = 0$, hence a real root exists. To prove uniqueness, note that $f'(x) = 3x^2 + 1 > 0$ for all $x \in \mathbb{R}$, so f is strictly increasing. A strictly increasing continuous function can cross the horizontal axis at most once, so the root is unique.

Solution 3(c). Set $m = \min_{[a,b]} f$ and $M = \max_{[a,b]} f$, which exist by continuity. Then

$$m(b - a) \le \int_a^b f(x)\,dx \le M(b - a).$$

Let

$$A := \frac{1}{b - a} \int_a^b f(x)\,dx.$$

Then $m \le A \le M$. If $A = m$ or $A = M$, we have $f(x) \equiv A$ for all $x \in [a, b]$, and we can take any $c \in (a, b)$. Otherwise, $m < A < M$. Since f is continuous on $[a, b]$, there exist points $x_1, x_2 \in [a, b]$ such that $m = f(x_1) < A < f(x_2) = M$. Thus, the Intermediate Value Theorem implies that there exists $c \in (x_1, x_2) \subset (a, b)$ such that $f(c) = A$.

4. (20 pts) Riemann integral and fundamental theorem of calculus.

Let $f : [a, b] \to \mathbb{R}$ be differentiable and let $\epsilon > 0$ and $\epsilon < \frac{b-a}{2}$. For $x \in [a + \epsilon, b - \epsilon]$, define

$$g_\epsilon(x) := \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} f(t)\, dt,$$

and $g_\epsilon(x) = 0$ for $x \notin [a + \epsilon, b - \epsilon]$.
(a) (7 pts) Show that g_ϵ is differentiable on $[a + \epsilon, b - \epsilon]$.
(b) (7 pts) Evaluate $\lim_{\epsilon\to 0} g'_\epsilon(x)$ for $x \in (a, b)$.
(c) (6 pts) Evaluate $\lim_{\epsilon\to 0} g_\epsilon(x)$.

Solution 4(a). Since f is differentiable on $[a, b]$, it is continuous on $[a, b]$. Thus, by the Fundamental Theorem of Calculus, $g_\epsilon(x) = \frac{1}{2\epsilon}[\int_{x-\epsilon}^{\frac{a+b}{2}} f + \int_{\frac{a+b}{2}}^{x+\epsilon} f]$ is differentiable on $[a + \epsilon, b - \epsilon]$ and

$$g'_\epsilon(x) = \frac{1}{2\epsilon}\big(f(x + \epsilon) - f(x - \epsilon)\big).$$

(b). For $x \in (a, b)$, we have

$$\lim_{\epsilon\to 0} g'_\epsilon(x) = \lim_{\epsilon\to 0} \frac{f(x + \epsilon) - f(x - \epsilon)}{2\epsilon} = f'(x),$$

where the last equality follows from the definition of the derivative of f at x.

(c). For $x \in (a, b)$, the continuity of f implies that for any $\delta > 0$, there exists $\eta > 0$ such that

$$|f(t) - f(x)| < \delta, \quad \text{whenever } |t - x| < \eta.$$

For any $\epsilon < \eta$, we have

$$|g_\epsilon(x) - f(x)| = \left|\frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} (f(t) - f(x))\, dt\right| \le \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} |f(t) - f(x)|\, dt < \delta.$$

Since $\delta > 0$ is arbitrary, we conclude that

$$\lim_{\epsilon\to 0} g_\epsilon(x) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} f(t)\, dt = f(x).$$

5. (20 pts) Sequences of functions and uniform convergence.

Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = x^n \qquad (n = 1, 2, 3, \dots).$$

(a) (6 pts) Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in [0,1]$.

(b) (8 pts) Decide whether $f_n \to f$ uniformly on $[0,1]$. Justify your answer.

(c) (6 pts) Compute $\int_0^1 f_n(x)\,dx$ for each n and compute $\lim\limits_{n\to\infty} \displaystyle\int_0^1 f_n(x)\,dx$.

Solution 5(a). For $x \in [0,1)$ we have $|x| < 1$ and hence $x^n \to 0$ as $n \to \infty$. For $x = 1$ we have $f_n(1) = 1$ for all n. Thus

$$f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases}$$

Solution 5(b). The convergence is not uniform on $[0,1]$. Suppose for contradiction that $f_n \to f$ uniformly. Then

$$\|f_n - f\|_\infty := \sup_{x\in[0,1]} |f_n(x) - f(x)| \to 0.$$

But

$$\sup_{x\in[0,1)} |f_n(x) - 0| = \sup_{x\in[0,1)} x^n = 1,$$

and $|f_n(1) - f(1)| = |1 - 1| = 0$, so in fact

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1$$

for every n. Hence the sup norm does not tend to 0, a contradiction. Therefore f_n does not converge uniformly to f on $[0,1]$.

Solution 5(c). For each n,

$$\int_0^1 f_n(x)\,dx = \int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}\bigg|_0^1 = \frac{1}{n+1}.$$

Hence $\lim\limits_{n\to\infty} \displaystyle\int_0^1 f_n(x)\,dx = 0.$

On the other hand, $f(x) = 0$ for all $x \in [0,1)$ and $f(1) = 1$, so $\int_0^1 f(x)\,dx = 0$. Thus in this example

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx,$$

even though the convergence is not uniform.

Theorem 1 (Dini, monotone nonnegative version) *Let (f_n) be a sequence of continuous functions on $[a, b]$ such that*

1. *$f_n(x) \geq f_{n+1}(x) \geq 0$ for all $x \in [a, b]$ and all n (monotone decreasing, nonnegative);*
2. *$f_n(x) \to 0$ for each $x \in [a, b]$ (pointwise convergence).*

Then $f_n \to 0$ uniformly on $[a, b]$.

Proof. For each n, define

$$s_n := \sup_{x \in [a,b]} f_n(x) = f_n(x_n),$$

where $x_n \in [a, b]$ because f_n is continuous on the closed interval $[a, b]$ so the supremum is attained.

From the pointwise monotonicity $f_n(x) \geq f_{n+1}(x)$, we have

$$s_n = \sup_x f_n(x) \geq \sup_x f_{n+1}(x) = s_{n+1}.$$

Thus $\{s_n\}$ is a decreasing sequence of nonnegative real numbers, hence convergent $s_n \downarrow s$ for some $s \geq 0$.

Our goal is to show $s = 0$. Suppose, for contradiction, that $s > 0$.

Step 1: Extract a cluster point of $\{x_n\}$. The sequence $\{x_n\}$ lies in the bounded interval $[a, b]$, so there exists a subsequence $\{x_{n_k})\}$ and a point $x_0 \in [a, b]$ such that $x_{n_k} \to x_0 \quad$ as $k \to \infty$.

Along this subsequence, we have

$$s_{n_k} = f_{n_k}(x_{n_k}) \to s.$$

Step 2: Show $f_m(x_0) \geq s$ for all m. Fix an arbitrary integer $m \in \mathbb{N}$. For all k with $n_k \geq m$, the monotonicity $f_m \geq f_{n_k}$ implies

$$f_{n_k}(x) \leq f_m(x) \quad \text{for all } x \in [a, b],$$

and in particular at x_{n_k}:

$$s_{n_k} = f_{n_k}(x_{n_k}) \leq f_m(x_{n_k}).$$

Sending $k \to \infty$ and using the fact that f_m is continuous and $x_{n_k} \to x_0$, we have

$$s = \lim_{k \to \infty} s_{n_k} \leq f_m(x_0) = \lim_{k \to \infty} f_m(x_{n_k}).$$

As m is arbitrary, we conclude that

$$f_m(x_0) \geq s \quad \text{for all } m \in \mathbb{N},$$

which contradicts with the pointwise convergence that $f_n(x_0) \to 0$ as $n \to \infty$.

Therefore, we must have $s = \lim_{n \to \infty} s_n = 0$, and hence $f_n \to 0$ uniformly on $[a, b]$.