MATH 405 FALL HOMEWORK SOLUTIONS 6

DANIEL PEZZI

Note that there may be typos and errors in these solutions. Consider it an exercise to find them!

Homework 6

Chapter 2.5 Problem 4

(1) Define $S_N = \sum_{n=1}^{\infty} x_n$. To say the sum converges is to say that S_N converges as a sequence.
Define $P_N = \sum_{n=1}^{N} x_{2n} + x_{2n+1}$. Note we have

$$P_N = x_2 + x_3 + ... + x_{2N+1} = S_{2N+1}.$$

The convergence of P_N is now obvious.

(2) Consider $x_n = (-1)^n$. The sum with two terms is always 0, but clearly this sum diverges as the limit is not 0.

Chapter 2.5 Problem 6

(1) In the proof of the ratio test, convergence is only used to show that for some N we have, for $r > 0$ and every $n \geq N$, that

$$\frac{|x_{n+1}|}{|x_n|} < r.$$

This is what we assume in the stronger formulation. The rest of the proof is the same.

(2) Note that there exists an N such that $|x_{n+1}| \geq |x_n|$. Note implicitly the condition listed implies $|x_n| \neq 0$ for every $n \geq N$. This implies the sequence x_n does not converge to 0, as x_{n+1} must be further from 0 than x_n and induction implies the entire sequence is therefore at least $|x_N|$ form 0.

Chapter 2.5 Problem 11

The listed condition implies that there exists an $N > 0$ and $c, C > 0$ such that for $n \geq N$, we have

$$c < \frac{b_n}{a_n} < C.$$

To see this, suppose this was not true. That is, for every N and $C > 0$, we could fine a_n, b_n such that $\frac{b_n}{a_n} \geq C$ with $n \geq N$. Construct a subsequence such that $\frac{b_{n_k}}{a_{n_k}} \geq k$. This contradicts the assumption. The case for c is similar.

This implies $ca_n < b_n < Ca_n$. The conclusion follows as

$$c \sum a_n \leq \sum b_n \leq C \sum a_n.$$

Chapter 2.5 Problem 16

As convergence is only dependent on the tails, it suffices the series only satisfy the criterion for large n. Applying the criterion, we wish to test the convergence of

$$\sum \frac{n}{2^n}, \sum \frac{1}{n \ln(2)}, \sum \frac{1}{\ln(2)^2 n^2}, \sum \frac{1}{n \ln(2) \ln(n \ln(2))}.$$

The first sum converges as $n < 2^{.5n}$ for large n and the geometric series test. The second series diverges because it is a harmonic series. The second converges by the p-test. The fourth diverges because it of the same form as (b) before we applied the criterion.

Chapter 2.6 Problem 1

(1) Clearly $n^2 < 2^{2n+1}$ so $\frac{1}{2^{2n+1}} < \frac{1}{n^2}$, so this series converges by the comparison test.
(2) This series diverges as $(-1)^n \frac{n-1}{n}$ does not approach 0.
(3) Converges by alternating series test.
(4) $\frac{n^n}{(n+1)^{2n}} \leq \frac{1}{n^n}$. Clearly $n^n > n^2$ and so by the comparison test this converges.

Chapter 2.6 Problem 5 abcd

(1) We have limsup$|2^n|^{1/n} = 2$, so the radius of convergence is $1/2$.
(2) This is 1 by exercise 2.6.10.
(3) By the ratio test we have $((n+1)!x^{n+1}/(n!x^n)) = (n+1)x$ which diverges for every x in the limit.
(4) This converges everywhere by comparing with Example 2.6.7. The coefficients are strictly smaller.

Chapter 2.6 Problem 10

(1) Expanding $(1+b_n)^n$ gives terms with a factor of b_n^2. This term occurs n Choose 2 times as each term has n factors and we want exactly two of them to be b_n. This gives that

$$(1+b_n)^n = \frac{n(n-1)}{2}b_n^2 + positiveterms.$$

As every term in the expansion is positive, removing them only makes the sum smaller. This gives the desired inequality.
Note that $(1+b_n)^n = n$, but $n(n-1) = n^2 - n$. As quadratics grow faster than linear terms, the only way for inequality to hold is if b_n is approaching 0.

(2) Apply Proposition 2.6.11 directly, using $\sup(|a_n b_n|) \leq \sup(|a_n|)\sup(|b_n|)$.

Chapter 2.6 Problem 13

Consider $x_n = (-1)^n \frac{1}{\sqrt{n}}$. By the alternating series test this converges, but the squared series is the harmonic series which diverges.