MATH 405 FALL HOMEWORK SOLUTIONS 6-12

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Homework 7

Chapter 3.1 Problem 1 a, d, e

a) We claim the limit is $\sqrt{c}$. First assume $x > 0$. For this we compute

$$|\sqrt{x} - \sqrt{c}| = |\frac{x-c}{\sqrt{x}+\sqrt{c}}|$$

Let $\epsilon > 0$. Suppose $|x - c| \leq \delta(\epsilon)$. Assume $c > 0$. We shall restrict δ small enough so that $\sqrt{c}/2 < \sqrt{x} + \sqrt{c}$. We this assumption, we may bound

$$|\frac{x-c}{\sqrt{x}+\sqrt{c}}| \leq 2\sqrt{c}^{-1}|x-c| \leq \delta 2\sqrt{c}.$$

We may now choose δ depending on ϵ and c to conclude. Now we must handle the case where $c = 0$. This case is simple. Choose $\delta < \epsilon^2$.

d) We approach this by changing variables. The limit exists if and only if the following two limits exist and are equal.

$$\lim_{x\to\infty} \sin(x)\cos(x), \lim_{x\to-\infty} \sin(x)\cos(x)$$

This is clearly not the case as sin and cos are both periodic and take more than one value on their fundamental domain $[0, 2\pi]$. This allows one to easily construct two sequences x_n, y_n that approach ∞ but such that $X = \cos(x_n)\sin(x_n) \neq \sin(y_n)\cos(y_n) = Y$ for every n. Therefore, the limit does not exist.

e) We claim the limit is 0. Indeed

$$|\sin(x)\cos(1/x)| \leq |\sin(x)|$$

As $\lim_{x\to 0}\sin(x) = 0$, the conclusion follows from some simple computations.

Chapter 3.1 Problem 10

Suppose there exists x, y such that $f(x) \neq f(y)$. Consider the sequence $(x, y, x, y, x, y, ...)$. The conclusion follows as this sequence evaluated at f does not converge.

Chapter 3.1 Problem 13

By definition of a cluster point we have the existence of a sequence $x_n \to c$ with $x_n \in S$ and where every element of the sequence appears at most once. Note that then $\{f(x_n)\}_{n=1}^{\infty}$ is a bounded subsequence, and therefore has a convergent subsequence by Bolzano-Weierstrass. Denote this sequence $\{f(x_k)\}$. Then x_k satisfies the requirements of the problem. It must converge to c because it is the subsequence of a convergent sequence.

Chapter 3.2 Problem 5

f is clearly continuous when $x \neq 0$ because it is the product of two continuous functions. We must therefore show $\forall \epsilon > 0$ there exists a $\delta > 0$ such that $|x| \leq \delta \implies |x \sin(1/x)| < \epsilon$. This is immediate as

$$|x \sin(1/x)| \leq |x| < \delta$$

So $\delta = \epsilon$ works.

Chapter 3.2 Problem 15

By definition of continuity there exists a $\delta(\epsilon) > 0$ such $|(x-y) - 0| < \delta \implies |g(x-y) - g(0)| < \epsilon$. Note that we are viewing $x - y$ here as a single small value close to 0. Then $|f(x) - f(y)| < |g(x-y) - g(0)| < \epsilon$ whenever $|x - y| < \delta$, so f is continuous (and actually uniformly so).

Chapter 3.2 Problem 17

It suffices to prove the result for the maximum because $-f(x)$ and $-g(x)$ are continuous. Fix $x \in \mathbb{R}$. Let $f(x) = M$ and $g(x) = N$. Then we have two cases: either $M \neq N$ or $M = N$.

Suppose $M = N$ and let $\epsilon > 0$. The there exists $\delta_1, \delta_2 > 0$ such that

$$|x - y| < \delta_1 \implies |f(y) - M| < \epsilon, |x - y| < \delta_2 \implies |g(y) - N| < \epsilon$$

In this case, take $\delta = \min\{\delta_1, \delta_2\}$. Then as $\max\{f(y), g(y)\}$ is either $f(y)$ or $g(y)$ at any given y, we are done as in either case the function is within ϵ of $M = N$.

Suppose $M \neq N$. Assume without loss of generality that $M > N$. Let $\epsilon > 0$ be smaller than $\frac{1}{2}(M+N)$. There exists a δ_1 such that $|f(y) - M| < \epsilon$ when $|y - x| < \delta_1$. There also exists a δ_2 such that $|g(y) - N| < \frac{1}{2}(M+N)$. Let $\delta = \min\{\delta_1, \delta_2\}$. $|x - y| < \delta \implies \max\{f(x), g(x)\} = f(x)$ and the result follows by continuity of $f(x)$. For larger ϵ, we may select the smaller δ as needed.

Chapter 3.3 Problem 2

Let $f(x) = x$ when $0 < x < 1$ and $f(x) = \frac{1}{2}$ when $x = 0, 1$. The desired properties are easily verified.

Chapter 3.3 Problem 4

First, some reductions. As $\sin(1/x)$ is only discontinuous at $x = 0$, we only need to prove the intermediate value property near 0, that is, in an interval (a, b) that contains 0. Additionally, we only need to prove it on the interval $[0, b)$ for a small a because of the periodicity of $\sin(x)$; if we selected a, b with $a < 0$ and $b > 0$, we can always make the problem more difficult by looking for a solution in $[0, b)$ instead of (a, b)

Let C be a value in the range of $\sin(1/x)$. We shall show there exists $c \in [0, b)$ such that $f(c) = C$. By our previous reductions this is enough to conclude. Note if $C = 0$ this is easy as $\sin(x) = 0$ when $x = k\pi$ and so $\sin(1/x) = 0$ when $x = \frac{1}{k\pi}$, and so by the Archimedean property we will eventually be able to find a zero is $(0, b)$ for every b. This strategy also works for other numbers. As $\sin(x)$ is periodic, for every C in its range, there exists $x_0 + 2\pi k$ such that $\sin(x_0 + 2\pi k) = C$ for every $k \in \mathbb{N}$. This gives us a sequence $y_k = \frac{1}{x_0+2\pi k}$ that approaches 0 such that $\sin(1/y_k) = C$. The result follows.

Chapter 3.3 Problem 10

Note $g(x) = f(x) - x$ is continuous. $f(x)$ having a fixed point is equivalent to $g(x)$ having a 0. Note $g(0) \geq 0$ and $g(1) \leq 0$, by the domain of f. As g is continuous, we may apply the Intermediate Value Theorem to conclude.

Note: this trick may seem dumb, but it quite common in math. When trying to find a specific point with a given property, it is useful to turn the problem into finding the zero of a function as we have many tools to do so.

Homework 8

Chapter 3.4 Problem 8

We will show that $\exists \epsilon > 0$ such that $\forall \delta > 0$, there exists x, y such that $|f(x) - f(y)| > \epsilon$. Take $\epsilon = 1/2$ (many others work). Let $x_n \to \infty$ be such that $\sin(x_n) = 1$ and let $y_n \to \infty$ be such that $\sin(y_n) = 0$ for every n. Let $\delta > 0$. Then there exists N, M such that $x_N, y_M > 10\delta^{-1}$. We then have

$$|\sin(x_N^{-1}) - \sin(y_M^{-1})| = 1 > \epsilon.$$

But also $|x_N^{-1} - y_M^{-1}| < \frac{1}{2}\delta$ and so we are done.

Chapter 3.4 Problem 14

Suppose $\exists x_0$ such that $|f(x_0)| > K/2$. Then x_0 is within $1/2$ of either 0 or 1. Assume without loss of generality that x_0 is within $1/2$ of 0 (otherwise reflect around $x = \frac{1}{2}$). Then

$$|f(x_0) - f(0)| = |f(x_0)| \leq K|x_0 - 0| \leq \frac{K}{2}.$$

This is a contradiction and we are done. For the saturating example take $f(x) = x$ when $0 \leq x \leq 1/2$ and $f(x) = 1 - x$ when $\frac{1}{2} \leq x \leq 1$.

Chapter 3.4 Problem 15

Clearly f is uniformly continuous in $[-10P, 10P]$ because the domain is a compact interval. Let $\epsilon > 0$. Let δ be the constant so that $x, y \in [-10P, 10P]$ and $|x - y| < \delta$ implies that $f|(x) - f(y)| < \epsilon$.

Fix $\epsilon > 0$. Let $\delta' = \min\{\frac{P}{2}, \delta\}$. Then if $x, y \in \mathbb{R}$ are such that $|x-y| < \delta'$, we can translate them by some integer value of P to get x_{shift}, y_{shift} such that $x_{shift}, y_{shift} \in [-2P, 2P] \subset [-10P, 10P]$. Additionally, the function value is unchanged as we have shifted by the period. The restriction that x, y are at least within $\frac{P}{2}$ of each other makes this possible. By the assumption of uniform continuity on $[-10P, 10P]$ and periodicity, we have

$$|f(x) - f(y)| = |f(x_{shift} - f(y_{shift}))| < \epsilon,$$

which is uniform in x, y.

Chapter 3.5 Problem 9

(1) We take $\varphi(x) = \frac{-2x}{x^2-1}$. The easiest way to compute this is to guess something of the form $\frac{p(x)}{x^2-1}$ as an asymptote must occur at $x = -1, 1$.

(2) It is useful to perform a change of variables to write

$$g(\varphi(y)) = f(y).$$

Note that $\varphi(y)$ is continuous, and so g being continuous implies f is continuous on the domain of φ, which is $(-1, 1)$. Note this interval is the length of the period of f, so for f to be continuous on $[-1, 1]$ we need the listed condition, as when $x \to 1$ we have $\varphi(x) \to \infty$, and when $x \to -1$ we have $\varphi(x) \to -\infty$. The other direction is similar.

Chapter 3.6 Problem 3

If I is an open interval there is nothing to prove. WLOG assume f is increasing at it approaches an endpoint a (this can be the left or right endpoint as reflections of continuous functions are continuous). As f is monotonically increasing, and $f(a)$ is finite, by monotone convergence theorem $\lim_{x \to a} f(x)$, where the limit is one sided, exists. For f to not be continuous, $f(a) \neq \lim_{x \to a} f(x)$. By monotonicity there can be no values of $f(I)$ in between, which means the image is not an interval. So by the contrapositive we are done.

Chapter 4.1 Problem 6

We wish to compute

$$\lim_{x\to 0}\frac{\sin(x)}{x}.$$

Intuitively we expect this to be 1. As such, compute

$$\lim_{x\to 0}|\frac{\sin(x)}{x}-1| = \lim_{x\to 0}|\frac{\sin(x)-x}{x}| \leq \lim_{x\to 0}|x| = 0$$

So the result is proved.

Chapter 4.1 Problem 10

We have

$$x = g(f(x)) \implies 1 = g'(f(x))f'(x) \implies g'(f(x)) = \frac{1}{f'(x)}.$$

Chapter 4.1 Problem 11

We compute directly

$$\lim_{x\to c}\frac{f(x)g(x)-f(c)g(c)}{x-c}.$$

By assumption we have $g(c) = 0$. This reduces to

$$\lim_{x\to c}\frac{f(x)g(x)}{x-c}.$$

We conjecture this limit is 0. Assume $|f(x)| \leq M$. Then we compute

$$|\frac{f(x)g(x)}{x-c}| \leq M|\frac{g(x)}{x-c}|.$$

By taking $x \to c$ we can make $g(x)/(x-c)$ approach 0, using the assumption that $g(x)$ has 0 derivative at $x = c$. The proof concludes and $h(x)$ is differentiable at $x = c$.

Homework 9

Chapter 4.2 Problem 4

This is a straightforward application of the Mean Value Theorem. As $\lim_{h\to 0} h^{-1}(f(c+h)-f(c)) = f'(c)$, we can select a sequence of points indexed by h_n such that the difference quotient approaches $f'(c)$ as h_n approaches 0. The Mean Value Theorem assures us a sequence of values of the derivative approaching the derivative as well. This does not imply continuity as we only have convergence for one sequence, not every sequence.

Chapter 4.2 Problem 5

We compute the derivative of f

$$|f'(x)| = |\lim_{y\to x} \frac{f(x)-f(y)}{x-y}| \leq \lim_{x\to y} |x-y| = 0.$$

So the derivative is 0. The implies the function is constant by Mean Value Theorem.

Chapter 4.2 Problem 9

As $g'(c)$ is positive, $g(x) \neq 0$ when x is near c. Otherwise, by Mean Value Theorem, we would find a point with 0 derivative close to c which contradicts positivity as g' is continuous.

By inserting $f(c) = g(c) = 0$, we have that $\frac{f(x)-f(c)}{g(x)-g(c)}$ satisfies the assumptions of Theorem 4.2.5, after some rearranging, which gives the previous fraction is equal to $\frac{f'(\xi)}{g'(\xi)}$ for some $\xi \in [c-|x-c|, c+|x-c|]$. The result follows from taking limits as $x \to c$ which forces $\xi \to c$.

Chapter 4.2 Problem 10

There exists a sequence x_n such that $f(x_n) \to \infty$. The boundedness of the domain means the difference quotient between the points x_1 and x_n is unbounded. MVT gives the result.

Chapter 4.3 Problem 7

We guess $f(x) = \frac{1}{2}ax^2 + bx + c$. It is easily verified that this function has the desired properties.

Assume there is a g such that the same statements hold. Then g and f must have the same second derivatives everywhere. As they have the value of the derivative at 0, the MVT implies they have the same first derivative everywhere as

$$\frac{g'(x)-g'(0)}{x} = g''(c) = f''(c) = \frac{f'(x)-f'(0)}{x}.$$

The same argument again shows, using $f(0) = g(0)$, that f and g have the same function values for all $x \neq 0$, and we are done

Chapter 4.3 Problem 10

For both cases, write out Taylor's theorem.

$$f(x) = \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x).$$

The remainder term looks like a constant times $(x-x_0)^{n+1}$. Because it is a larger power than the main terms which is like $(x-x_0)^n$, for x small enough the remainder is smaller than the main term. In particular, this is true as x approaches x_0. Using the explicit form of the remainder, apply a derivative test to conclude, using the assumption that $f^n(x_0)$ is not zero. Classify based on the parity.

Chapter 4.4 Problem 1

Write $\mathbb{R}$ as the union of overlapping open intervals. The inverse function theorem gives the result on any interval. The overlap forces the function to be defined over all of $\mathbb{R}$, as the given functions must agree on any overlap.

Chapter 4.4 Problem 8

Both of these follow from $x = f(g(x))$ and writing out the proof of the chain rule.

Homework 10

Chapter 5.1 Problem 3

We have

$$L(P_k, f) \leq \sup L(P, f) \leq \inf U(P, f) \leq U(P_k, f).$$

The result follows.

Chapter 5.1 Problem 5

We define a partition P_k by $[-1, -\frac{1}{k}] \cup [-\frac{1}{k}, \frac{1}{k}] \cup [\frac{1}{k}, 1]$. It is easily computed that

$$U(P_k, f) = 1 + \frac{1}{k}, L(P_k, f) = 1 - \frac{1}{k}.$$

Applying the previous exercise gives the result with the integral being 1.

Chapter 5.1 Problem 11

a) Let P_n be the partition defined by $1/n$ spacing between bounds. Clearly $L(P_n, f) \leq R_n \leq U(P_n, f)$. Note that $L(P_n, f)$ and $U(P_n, f)$ are monotone in n.

As f is Riemann integrable there exists Q_k a sequence of partitions such that $U(Q_k, f)$ approaches $\int f$. For each partition there is a minimum distance d_k that separates two points. Let N_k be the smallest integer such that $N_k^{-1} \leq d_k/100$. Then

$$U(P_{N_k}, f) \leq U(Q_k, f).$$

By taking limits and using the squeeze theorem, this constructs a subsequence of $U(P_n, f)$ that approaches the integral. But the sequence is monotone, so the entire sequence also converges. Therefore

$\lim_{n\to\infty} R_n \le \int f$. A similar argument for the lower bound gives equality.

b) Consider Dirichlet's function that is 1 on the rationals and 0 everywhere else.

Chapter 5.1 Problem 15

This follows from the definition of lower and upper sum, as the sup and inf are taken over the closed set $[x_i, x_{i+1}]$, so there is overlap between adjacent intervals. Proving the function is constant on an interval is trivial at this point as if the function is not constant, there must exists two adjacent intervals with different values, which is a contradiction.

Chapter 5.2 Problem 10

That $|f(x)|$ is bounded with finitely many discontinuities is immediate from f being bounded and the absolute value function being continuous. Note that for every partition

$$U(P, f) \le U(P, |f|)$$

as $f(x) \le |f(x)|$ for every x. This implies
$\int_a^b f(x)dx \le \int_a^b |f(x)|dx$. Taking absolute values gives the result.

Chapter 5.2 Problem 11

Fix a large number k. Our partition will be in two cases. For each $1 \le i \le k$, there are at most i point in $[0, 1]$ such that $f(x) = i^{-1}$. Around these points put an interval of length k^{-2}. Partition the rest of the interval using intervals of length around k^{-1}.

For the intervals of length k^{-1}, there are at most k of them, and the maximum height of the function is k^{-1}. This gives a contribution to the upper sum of $\sim k^{-1}$. For the intervals of length k^{-2}, their contribution is at most

$$\sum_{i=1}^{k} \frac{1}{k^2} i i^{-1} = \frac{1}{k}.$$

Adding the two and taking the limit gives the upper integral as 0. The lower integral is obviously 0 so we conclude.

Chapter 5.2 Problem 14

a) Clearly the function is bounded, so the upper and lower integrals are bounded. By monotonicity both converge. We need to show they converge to the same value.

Consider a partition of length k^{-1}. As the function is increasing, the rightmost point is always the supremum and the leftmost point is always the infimum. As we have taken a uniform distribution, the Upper and Lower sum therefore only differ over the subintervals $[a, a+k^{-1}]$ and $[b-k^{-1}, b]$ because, for any other subinterval, we can match a value in the upper sum to one in the lower. But as $k \to \infty$ the error, because the function is bounded, goes to 0, and we are done.

b) Consider $-f$ and use linearity and the previous exercise.

c) $f, -g$ are obviously bounded, so the upper and lower integrals of their sum is the sum of their upper and lower integrals. The result follows.

Chapter 5.2 Problem 18

First we prove this for constant functions. Note that the integral of $\sin(nt)$ over $[\frac{k}{n}, \frac{k+2\pi}{n}]$ is 0 for every k. Given $[a, b]$, we can partition this into intervals of length exactly $2\pi/n$ except for at the endpoints, which will have length smaller than this value. We then have$|\int_a^b C\sin(nt)dt| \leq C4\pi/n$ by the triangle inequality. Taking the limit as $n \to \infty$ gives the result.

Fix $\epsilon > 0$. Then there exists a δ such that $|x-y| < \delta \implies |f(x)-f(y)| < \epsilon$. This δ is uniform as the interval is closed. This also implies we may write

$$f(x) = f(x_0) + g(x)$$

on any interval $[x_0, x_1]$ of length less than δ for a g such that $|g| < \epsilon$. Now choose n so that $2\pi/n$ is much smaller than δ. Applying the result for constant functions we get

$$|\int_a^b f(t)\sin(nt)dt| \leq M2\pi/n + \epsilon\frac{2\pi}{n}n \leq 2\pi Mn^{-1} + 2\pi\epsilon.$$

The term M is the maximum value of f on $[a, b]$ which is finite by continuity. Therefore, by further taking n large enough depending on ϵ, we have $x_n \leq 2\epsilon$ when n is large enough. As this holds for every ϵ we are done.

Homework 11

Chapter 5.3 Problem 1

Recall $\int_{-x}^{x} = \int_0^x - \int_0^{-} x$. This puts us in a position to use FTC. Compute

$$\frac{d}{dx}(\int_{-x}^{x} e^{s^2}) = e^{x^2} - (-e^{x^2}) = 2e^{x^2}.$$

The second negative comes from the chain rule on $-x$ or a change of variables. As a sanity check, the integrand is always positive so the derivative of the integral should always be positive as well.

Chapter 5.3 Problem 5

Apply FTC to $\frac{d}{dx}\Big(F(x)G(x)\Big)$ and subtract over one of the terms.

Chapter 5.3 Problem 7

a) By FTC we have

$$g'(x) = \frac{f(x+\epsilon) - f(x-\epsilon)}{2\epsilon}.$$

Note the similarity to 5.3.1.

b) Taking the $\lim_{\epsilon \to 0}$ gives $g'(x) = f'(x)$. This can be seen by adding and subtracting $f(x)$ in the numerator and applying limit rules.

c) This is best done geometrically. The graph of $|x|$ is piecewise graph of two lines, so the area under the curve is the area of parallelogram. Suppose $x \geq 1$. Then

$$g(x) = \frac{1}{2}\int_{x-1}^{x+1} y dy = \frac{1}{4}((x+1)^2 - (x-1)^2).$$

When $x \leq 1$ we similarly have

$$g(x) = \frac{1}{2}\int_{x-1}^{x+1} -y dy = \frac{1}{4}((x-1)^2 - (x+1)^2).$$

For the expression at $x = 0$, split the integral and compute each term. One will get an additional dependency on the sign of x.

Chapter 5.4 Problem 6

(1) The function $\frac{1}{x}$ is strictly decreasing, so its right hand Riemann sum is bounded by the integral, and its integral is bounded by its left hand Riemann sum. Writing out the formulas in this instance gives the result with $\Delta x = 0$.

(2) Viewing $\ln(n) = \int_1^n \frac{1}{x}$ and using the upper bound in the previous exercise, we get that the series is monotonically increasing. The lower bound in the previous exercise implies that it is bounded. Therefore, it converges by the monotone convergence theorem.

Chapter 5.5 Problem 5

Using the power rule, $\int x^{-1/2} dx = 2x^{1/2} + C$. Therefore it can be interpreted as an improper integral by FTC.

Chapter 5.5 Problem 9

Suppose $\lim_{x\to\infty} f(x) = a$. As f is decreasing, this implies $f(x) \geq a$ for all x. As f is positive, we have $\int_0^N f \geq aN$ for every $N \in \mathbb{R}^+$. If $\int_0^\infty f$ exists, this forces $a = 0$. To show the converse fails consider $f(x) = x^{-1}$.

Chapter 5.5 Problem 11

Consider the function that is 1 on $[0, 1/2)$ and -1 on $[1/2, 1)$. Extend this function to $\mathbb{R}$ by the relation $f(x) = f(x + 1)$. Clearly, $\int_0^n f(x) = 0$. But the integral is not defined when the domain is 0 to ∞.

Chapter 5.5 Problem 13

a)

$$p.v. \int_{-1}^{1} \frac{1}{x} dx$$
$$= \lim_{\epsilon\to 0^+} -\ln(|-1|) + \ln(|-\epsilon|) - \ln(\epsilon) + \ln(|1|) = \lim_{\epsilon\to 0^+} \ln(|\epsilon/\epsilon|) = 0.$$

b) From the previous calculation we get

$$\lim_{\epsilon\to 0^+} -\ln(|-1|) + \ln(\epsilon) - \ln(2\epsilon) + \ln(|1|) = \ln(\frac{1}{2}) \neq 0.$$

c) Integrable implies $\int_{-\epsilon}^{\epsilon} f(x)dx \to 0$ as $\epsilon \to 0$. The result follows.

d) This follows immediately by symmetry.

e)

$$\frac{f(x)}{x} = \frac{f(x) - f(0) + f(0)}{x} = \frac{f(x) - f(0)}{x} + \frac{f(0)}{x}.$$

The principal value of the first term exists by the assumption that f is differentiable (the limit as x approaches 0 is $f'(x)$). The principal value of the second term is zero by the previous part.

Homework 12

Chapter 6.1 Problem 2

(1) We have $\lim_{n\to\infty} \frac{x}{n} = 0$ for every x. By continuity this implies $\lim_{n\to\infty} n^{-1}e^{x/n} = 0$.

(2) No. For a fixed n, we can make the expression as large as we like by taking x arbitrarily large.

(3) Yes, as $|e^{x/n}| \leq 1$ uniformly in n and x on this set, so $|n^{-1}e^{x/n}| \leq n^{-1}$ which converges to 0.

Chapter 6.1 Problem 7

Take the limit of both sides. The right hand side is uniform in x so the left hand side is as well.

Chapter 6.1 Problem 11

$f(x)$ is a piecewise defined function where each component is continuous. This is continuous if and only if the functions agree at the transition points. This is obviously true in this case.

$f(x)$ is continuous on the interval $[a-1, b+1]$. The convergence is obviously uniform outside of this interval because $f(x)$ is constant. So f is uniformly continuous on $[a-1, b+1]$. Fix $\epsilon > 0$. Then there exists a $\delta > 0$ so that for all $x, y \in [a-1, b+1]$ such that $|x-y| < \delta$ we have

$$|f(x) - f(y)| < \epsilon.$$

Restrict to n large enough so that $\frac{2}{n} < \epsilon$. Then

$$|f_n(x) - f(x)| = |\frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{2}} f(s) ds - f(x)|.$$

The following is a surprisingly powerful trick. As $f(x)$ is constant with respect to the dummy variable s, we have

$$f(x) = \frac{2}{n} \frac{n}{2} f(x) = \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{2}} f(x) ds.$$

The result follows from the triangle inequality and our assumption that $\frac{2}{n} < \delta$.

Chapter 6.1 Problem 13

$$|f(x)| = |\lim_{n\to\infty} f_n(x)| = \lim_{n\to\infty} |f_n(x)| \leq B.$$

Chapter 6.2 Problem 2

Clearly $|x^n/n| \leq n^{-1}$ on $[0, 1]$, so the convergence is uniform. The function it converges to is 0. But $f_n'(x) = x^{n-1}$ which, evaluated at 1, gives 1 uniformly in n. So the limit of the derivatives is 1 but the limiting function has derivative 0. This is part of why we define differentiability on open sets.

Chapter 6.2 Problem 7

For each fixed x, define $f(x)$ to be the limit of the Cauchy sequence $f_n(x)$. The assumption on the C^1 norm is stronger than uniform convergence on x, as we sum the supremum of the function, which is what we need, and the supremum of the derivative. So the limit is immediately uniform. We also know that the sequence of derivatives is Cauchy and that the convergence is uniform by the same argument. An application of Theorem 6.2.10 gives the result.

Chapter 6.2 Problem 9

Let $g_n(x)$ be a function that is n^3 on the interval $[0, \frac{1}{100n^2}]$. The integral of this function is $\frac{1}{100}n$. In particular, the L^1 norm is unbounded in the limit. To construct a sequences of functions whose pointwise limit is 0, use the following rule. Let $f_1(x) = g_1$. Then, let $f_n(x)$ be $g_n(x)$ but horizontally shifted by some value such that

(1) The set of points where f_n is non-zero is contained in $[0, 1]$.
(2) For each fixed n, the set of points where $f_n(x)$ is non-zero does not intersect the set of points where f_k is non-zero for every $1 \leq k < n$.

The first property can easily be satisfied for some set of shifts. For the second, note that the size of the non-zero set for each function is $\frac{1}{100}n^{-2}$. By a p-series test, the total set of points where some $f_n(x)$ is non-zero is less than 1. In particular, we can always find a shift so that $f_n(x)$ satisfies property 2. This can be made a little more explicit with induction.

We have previously said the L^1 norm diverges. For each fixed x, there will be some N such that $n \geq N$ implies $f_n(x) = 0$. This shows the pointwise limit is 0 for every x.

Chapter 6.2 Problem 10

As f is continuous, we have $s_n = \sup_{[a,b]} f_n(x)$ exists for every n. Note that this sequence is decreasing. Let x_n be the point such that $f_n(x_n) = s_n$. As this sequence is bounded, it contains a convergent subsequence. Let x_m be such a subsequence and s_m the corresponding values. Let y be the limit of the x_m.

Fix $\epsilon > 0$. We now must be careful of our dependencies. Obviously the convergence will be uniform if we show for our subsequence

$$s_m \to 0.$$

Let $f_n(y)$ be the sequence generated from evaluating our family of functions at y, the limit point of the supremums. First, we prove a lemma. That is, the family f_n is uniformly continuous uniformly in n, up to a factor, when we are near y.

There exists N such that $n \geq N$ means $f_n(y) < \epsilon$. For fixed n, there exists a δ such that $|z - y| < \delta$ implies $|f_n(y) - f_n(z)| < \epsilon$. We then have

$$|f_n(z)| \leq |f_n(z) - f_n(y)| + |f_n(y)| < 2\epsilon,$$

for z close to y and n large enough. Note how nothing we have said depends on x. Suppose $|z - y| < \delta$ and $k > n$. Then by the same argument and the fact our sequence is pointwise decreasing we get

$$\begin{aligned}|f_k(x) - f_k(y)| &\leq |f_k(x)| + |f_k(y)| \\ &\leq |f_n(x)| + |f_n(y)| \\ &\leq 2\epsilon + 2\epsilon = 4\epsilon.\end{aligned}$$

Now we have enough to conclude. Recall we want to show s_m goes to zero. Fix $\epsilon > 0$. Fix N such that $n \geq N$ implies $f_n(y) < \epsilon$. Fix δ such that $|z - y| < \delta$ implies $|f_N(z) - f_N(y)| < \epsilon$. Lastly, select M such that $m \geq M$ implies $|x_m - y| < \delta$. Then we have

$$\begin{aligned}|s_m| &= |s_m + f_n(y) - f_n(y)| \\ &\leq |f_n(y)| + |s_m - f_n(y)| \\ &\leq |f_n(y)| + |s_m - f_m(y)| + |f_m(y) - f_n(y)| \\ &\leq \epsilon + 4\epsilon + 2\epsilon \\ &< 7\epsilon\end{aligned}$$

The first ϵ inequality comes from pointwise convergence. The second comes from uniform continuity uniformly in n up a factor for points close to y. We fixed n to apply such a proposition, possibly taking m to be smaller in response. The last follows from the triangle inequality and the assumption $m \geq n$.

As s_n is decreasing, we may pass this result up to the full sequence from the subsequence. $s_n \rightarrow 0$.